

Hölder-continuous rough paths by Fourier normal ordering

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We construct in this article an explicit geometric rough path over arbitrary d -dimensional paths with finite $1/\alpha$ -variation for any $\alpha \in (0, 1)$. The method may be coined as 'Fourier normal ordering', since it consists in a regularization obtained after permuting the order of integration in iterated integrals so that innermost integrals have highest Fourier frequencies. In doing so, there appear non-trivial tree combinatorics, which are best understood by using the structure of the Hopf algebra of decorated rooted trees (in connection with the Chen or multiplicative property) and of the Hopf shuffle algebra (in connection with the shuffle or geometric property). Hölder continuity is proved by using Besov norms.

The method is well-suited in particular in view of applications to probability theory (see the companion article [34] for the construction of a rough path over multidimensional fractional Brownian motion with Hurst index $\alpha < 1/4$, or [35] for a short survey in that case).

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Contents

0	Introduction	3
1	Iterated integrals : smooth case	8
1.1	From iterated integrals to trees	8
1.2	Permutation graphs and Fourier normal ordering for smooth paths	11
1.3	Tree Chen property and coproduct structure	13
1.4	Skeleton integrals	14

2	Regularization : the Fourier normal ordering step by step.	18
3	Proof of the geometric and multiplicative properties	22
3.1	Hopf algebras and the Chen and shuffle properties	23
3.2	Proof of the Chen and shuffle properties	26
4	Hölder estimates	34
4.1	Choice of the regularization scheme	35
4.2	A key formula for skeleton integrals	37
4.3	Estimate for the increment term	38
4.4	Estimate for the boundary term	41
5	Appendix. Hölder and Besov spaces	44

0 Introduction

Assume $t \mapsto \Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$, $t \in \mathbb{R}$ is a smooth d -dimensional path, and let $V_1, \dots, V_d : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be smooth vector fields. Then the classical Cauchy-Lipschitz theorem implies that the differential equation driven by Γ

$$dy(t) = \sum_{i=1}^d V_i(y(t)) d\Gamma_t(i) \quad (0.1)$$

admits a unique solution with initial condition $y(0) = y_0$. The usual way to prove this is to show by a functional fixed-point theorem that iterated integrals

$$y_n \mapsto y_{n+1}(t) := y_0 + \int_0^t \sum_i V_i(y_n(s)) d\Gamma_s(i) \quad (0.2)$$

converge when $n \rightarrow \infty$.

Assume now that Γ is only α -Hölder continuous for some $\alpha \in (0, 1)$. Then the Cauchy-Lipschitz theorem does not hold any more because one first needs to give a meaning to the above integrals, and in particular to the iterated integrals $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n)$, $n \geq 2$, $1 \leq i_1, \dots, i_n \leq d$.

The theory of rough paths, invented by T. Lyons [22] and further developed by V. Friz, N. Victoir [14] and M. Gubinelli [15] implies the possibility to solve eq. (0.1) by a *redefinition of the integration* along Γ , using as an essential ingredient a *rough path* $\mathbf{\Gamma}$ over Γ . By definition, a functional $\mathbf{\Gamma} = (\mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^N)$, $N = \lfloor 1/\alpha \rfloor$ = entire part of $1/\alpha$, is called a *rough path over* Γ if $\mathbf{\Gamma}_{ts}^1 = (\delta\Gamma)_{ts} := \Gamma_t - \Gamma_s$ are the two-point increments of Γ , and $\mathbf{\Gamma}^k = (\mathbf{\Gamma}^k(i_1, \dots, i_k))_{1 \leq i_1, \dots, i_k \leq d}$, $k = 1, \dots, N$ satisfy the following three properties:

- (i) (*Hölder continuity*) each component of $\mathbf{\Gamma}^k$, $k = 1, \dots, N$ is $k\alpha$ -Hölder continuous, that is to say, $\sup_{s \in \mathbb{R}} \left(\sup_{t \in \mathbb{R}} \frac{|\mathbf{\Gamma}_{ts}^k(i_1, \dots, i_k)|}{|t-s|^{k\alpha}} \right) < \infty$.
- (ii) (*multiplicative/Chen property*) letting $\delta\mathbf{\Gamma}_{tus}^k := \mathbf{\Gamma}_{ts}^k - \mathbf{\Gamma}_{tu}^k - \mathbf{\Gamma}_{us}^k$, one requires

$$\delta\mathbf{\Gamma}_{tus}^k(i_1, \dots, i_k) = \sum_{k_1+k_2=k} \mathbf{\Gamma}_{tu}^{k_1}(i_1, \dots, i_{k_1}) \mathbf{\Gamma}_{us}^{k_2}(i_{k_1+1}, \dots, i_k); \quad (0.3)$$

(iii) (geometric/shuffle property)

$$\mathbf{\Gamma}_{ts}^{n_1}(i_1, \dots, i_{n_1}) \mathbf{\Gamma}_{ts}^{n_2}(j_1, \dots, j_{n_2}) = \sum_{\mathbf{k} \in \text{Sh}(\mathbf{i}, \mathbf{j})} \mathbf{\Gamma}^{n_1+n_2}(k_1, \dots, k_{n_1+n_2}) \quad (0.4)$$

where $\text{Sh}(\mathbf{i}, \mathbf{j})$ is the set of shuffles of $\mathbf{i} = (i_1, \dots, i_{n_1})$ and $\mathbf{j} = (j_1, \dots, j_{n_2})$, that is to say, of permutations of $i_1, \dots, i_{n_1}, j_1, \dots, j_{n_2}$ which do not change the orderings of (i_1, \dots, i_{n_1}) and (j_1, \dots, j_{n_2}) .

There is a canonical choice for $\mathbf{\Gamma}$, called *canonical lift* of Γ , when Γ is a smooth path, namely, the iterated integrals of Γ of arbitrary order. If one sets

$$\mathbf{\Gamma}^{\text{cano}, n}(i_1, \dots, i_n) := \int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n), \quad (0.5)$$

then $\mathbf{\Gamma}^{\text{cano}} = (\mathbf{\Gamma}^{\text{cano}})_{n=1,2,\dots}$ satisfies properties (i), (ii), (iii) with $\alpha = 1$. Axiom (ii) receives a natural geometric interpretation in this case since $\mathbf{\Gamma}^{\text{cano}}$ measures the areas, volumes and so forth generated by $\Gamma^1, \dots, \Gamma^d$, see [14], while axiom (iii) may be deduced from Fubini's theorem. A further justification of axioms (i),(ii),(iii) comes from the fact that any rough path is a *limit in some sense of the iterated integrals of a sequence of smooth paths*, so $\mathbf{\Gamma}$ plays the rôle of a *substitute* of iterated integrals for Γ .

The problem we address here is the *existence and construction* of rough paths. It is particularly relevant when Γ is a random path; it allows for the pathwise construction of stochastic integrals or of solutions of stochastic differential equations driven by Γ . Rough paths are then usually constructed by choosing some appropriate smooth approximation Γ^η , $\eta \xrightarrow{\sim} 0$ of Γ and proving that the canonical lift of Γ^η converges in $L^2(\Omega)$ for appropriate Hölder norms to a rough path $\mathbf{\Gamma}$ lying above Γ (see [11, 32] in the case of fractional Brownian motion with Hurst index $\alpha > 1/4$, and [1, 18] for a class of random paths on fractals, or references in [23]).

A general construction of a rough path for deterministic paths has been given – in the original formulation due to T. Lyons – in an article by T. Lyons and N. Victoir [23]. The idea [14] is to see a rough path over Γ as a Hölder section of the trivial G -principal bundle over \mathbb{R} , where G is a free rank- N nilpotent group (or Carnot group), while the underlying path Γ is a section of the corresponding quotient G/K -bundle for some normal subgroup K of G ; so one is reduced to the problem of finding Hölder-continuous sections $g_t K \rightarrow g_t$. Obviously, there is no canonical way to do this in general. This

abstract, group-theoretic construction – which uses the axiom of choice – is unfortunately not particularly appropriate for concrete problems, such as the behaviour of solutions of stochastic differential equations for instance.

We propose here a new, *explicit method* to construct a rough path Γ over an arbitrary α -Hölder path Γ which rests on an *algorithm* that we call *Fourier normal ordering*. Let us explain the main points of this algorithm. The first point is the use of *Fourier transform*, \mathcal{F} ; Hölder estimates are obtained by means of Besov norms involving compactly supported Fourier multipliers, see Appendix. Assume for simplicity that Γ is compactly supported; this assumption is essentially void since one may multiply any α -Hölder path by a smooth, compactly supported function equal to 1 over an arbitrary large compact interval, and then restrict the construction to this interval. What makes the Fourier transform interesting for our problem is that $(\mathcal{F}\Gamma)'(\xi) = i\xi(\mathcal{F}\Gamma)(\xi)$ is a well-defined function; thus, the meaningless iterated integral $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n)$ is rewritten after Fourier transformation as some integral $\int_{-\infty}^{\infty} \dots \int_{-\infty}^{+\infty} f(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n$, where f is regular but not integrable at infinity along certain directions.

The second, main point is the *splitting of the Fourier domain of integration* \mathbb{R}^n into $\cup_{\sigma \in \Sigma_n} \mathbb{R}_{\sigma}^n$, Σ_n =set of permutations of $\{1, \dots, n\}$, where $\mathbb{R}_{\sigma}^n := \{|\xi_{\sigma(1)}| \leq \dots \leq |\xi_{\sigma(n)}|\}$, see section 2 for a more accurate definition involving the Besov dyadic decomposition. Away from the singular directions, the resulting integrals are naturally shown to have a polynomially decreasing behaviour at infinity implying the correct Hölder behaviour; simple examples may be read from [35]. However – as computations in section 4 clearly show, see also [35] for an elementary example – these bounds are naturally obtained only *after permuting the order of integration* by means of Fubini's theorem, so that the Fourier coordinates $|\xi_1|, \dots, |\xi_n|$ *appear in increasing order*. There appear in the process integrals over domains which differ from the simplex $\{t \geq t_1 \geq \dots \geq t_n \geq s\}$, which are particular instances of *tree integrals*, and that we call *tree skeleton integrals*.

The next step is to *regularize* the tree skeleton integrals so that Fourier integrals converge at infinity, *without* losing the Chen and shuffle properties (ii) and (iii). At this point it turns out to be both natural and necessary to re-interpret the above scheme in terms of tree Hopf algebra combinatorics. The interest for the study of Hopf algebras of trees or graphs surged out of a series of papers by A. Connes and D. Kreimer [8, 9, 10] concerning the mathematical structures hidden behind the Bogolioubov-Hepp-Parasiuk-Zimmermann (BPHZ) procedure for renormalizing Feynmann diagrams in quantum field theory [17], and is still very much alive, see for instance [20,

3, 4, 13, 6, 25, 36, 7], with applications ranging from numerical methods to quantum chromodynamics or multi-zeta functions or operads. It appears that the shuffle property may be stated by saying that *regularized skeleton integrals define characters* of yet another Hopf algebra called *shuffle algebra*, while the Chen property follows from the very definition of the regularized iterated integrals as a *convolution of regularized skeleton integrals*.

We show that the tree skeleton integrals may be regularized by integrating over appropriate subdomains of \mathbb{R}_σ^n avoiding the singular directions. The proof of properties (ii), (iii) uses Hopf combinatorics and does not depend on the choice of the above subdomains, while the proof of the Hölder estimates (i) uses both tree combinatorics and some elementary analysis relying on the shape of the subdomains.

It seems natural to look for a less arbitrary regularization scheme for the skeleton integrals. The idea of cancelling singularities by building iteratively counterterms, originated from the BPHZ procedure, should also apply here. We plan to give such a construction (such as dimensional regularization for instance) in the near future.

Let us state our main result. Throughout the paper $\alpha \in (0, 1)$ is some fixed constant and $N = \lfloor 1/\alpha \rfloor$.

Main theorem.

Assume $1/\alpha \notin \mathbb{N}$. Let $\Gamma = (\Gamma(1), \dots, \Gamma(d)) : \mathbb{R} \rightarrow \mathbb{R}^d$ be a compactly supported α -Hölder path. Then the functional $(\mathcal{R}\Gamma^1, \dots, \mathcal{R}\Gamma^N)$ defined in section 2 is an α -Hölder geometric rough path lying over Γ in the sense of properties (i), (ii), (iii) of the Introduction.

In a companion paper [34], we construct by the same algorithm an explicit rough path over a d -dimensional fractional Brownian motion $B^\alpha = (B^\alpha(1), \dots, B^\alpha(d))$ with arbitrary Hurst index $\alpha \in (0, 1)$ – recall simply that the paths of B^α are a.s. κ -Hölder for every $\kappa < \alpha$. The problem was up to now open for $\alpha \leq 1/4$ despite many attempts [11, 32, 33, 12]. Fourier normal ordering turns out to be very efficient in combination with Gaussian tools, and provides explicit bounds for the moments of the rough path, seen as a path-valued random variable.

The above theorem extends to paths Γ with finite $1/\alpha$ -variation. Namely (see [23], [21] or also [14]), a simple change of variable $\Gamma \rightarrow \Gamma^\phi := \Gamma \circ \phi^{-1}$ turns Γ into an α -Hölder path, with ϕ defined for instance as $\phi(t) := \sup_{n \geq 1} \sup_{0=t_0 \leq \dots \leq t_n=t} \sum_{j=0}^{n-1} \|\Gamma(t_{j+1}) - \Gamma(t_j)\|^{1/\alpha}$. The construction of the above Theorem, applied to Γ^ϕ , yields a family of paths with Hölder regularities $\alpha, 2\alpha, \dots, N\alpha$ which may alternatively be seen as a G^N -valued α -Hölder

path Γ^ϕ , where G^N is the Carnot free nilpotent group of order N equipped with any subadditive homogeneous norm. Then (as proved in [23], Lemma 8) $\Gamma := \Gamma^\phi \circ \phi$ has finite $1/\alpha$ -variation, which is equivalent to saying that Γ^n has finite $1/n\alpha$ -variation for $n = 1, \dots, N$, and lies above Γ .

Corollary.

Let $\alpha \in (0, 1)$ and $\alpha' < \alpha$. Then every α -Hölder path Γ may be lifted to a strong α' -Hölder geometric rough path, namely, there exists a sequence of canonical lifts $\Gamma^{(n)}$ of smooth paths $\Gamma^{(n)}$ converging to $\mathcal{R}\Gamma$ for the sequence of α' -Hölder norms.

The set of *strong* α -Hölder geometric rough paths is strictly included in the set of general α -Hölder geometric rough paths. On the other hand, as we already alluded to above, a weak α -Hölder geometric rough path may be seen as a strong α' -Hölder geometric rough path if $\alpha' < \alpha$. This accounts for the loss of regularity in the Corollary (see [14] for a precise discussion). The proviso $1/\alpha \notin \mathbb{N}$ in the statement of the main theorem is a priori needed because otherwise $\mathcal{R}\Gamma^N$ may not be treated in the same way as the lower-order iterated integrals (although we do not know if it is actually necessary). However, if $1/\alpha \in \mathbb{N}$, all one has to do is replace α by a slightly smaller parameter α' , so that the Corollary holds even in this case.

Note that the present paper gives unfortunately no explicit way of approximating $\mathcal{R}\Gamma$ by *canonical lifts* of smooth paths, i.e. of seeing it concretely as a *strong* geometric rough path. The question is currently under investigation in the particular case of fractional Brownian motion by using constructive field theory methods. Interestingly enough, the idea of controlling singularities by separating the Fourier scales according to a dyadic decomposition is at the core of constructive field theory [27].

Here is an outline of the article. A thorough presentation of iterated integrals, together with the skeleton integral variant, the implementation of Fourier normal ordering, and the extension to tree integrals, is given in section 1, where Γ is assumed to be *smooth*. The *regularization algorithm* is presented in section 2; the regularized rough path $\mathcal{R}\Gamma$ is defined there for an arbitrary α -Hölder path Γ . The proof of the Chen and shuffle properties is given in section 3, where one may also find two abstract but more compact reformulations of the regularization algorithm, see Lemma 3.5 and Definition 3.7. Hölder estimates are to be found in section 4. Finally, we gathered in an Appendix some technical facts about Besov spaces required for the construction.

Notations. We shall denote by \mathcal{F} the *Fourier transform*,

$$\mathcal{F} : L^2(\mathbb{R}^l) \rightarrow L^2(\mathbb{R}^l), f \mapsto \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} f(x) e^{-i\langle x, \xi \rangle} dx. \quad (0.6)$$

Throughout the article, $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^d$ is some compactly supported α -Hölder path; sometimes, it is assumed to be smooth. The *permutation group* of $\{1, \dots, n\}$ is denoted by Σ_n . Also, if $a, b : X \rightarrow \mathbb{R}_+$ are functions on some set X such that $a(x) \leq Cb(x)$ for every $x \in X$, we shall write $a \lesssim b$. Admissible cuts of a tree \mathbb{T} , see subsection 1.3, are usually denoted by v or w , and we write $(\text{Roo}_{\mathbf{v}}(\mathbb{T}), \text{Lea}_{\mathbf{v}}(\mathbb{T}))$ (*root part* and *leaves*) instead of the traditional notation $(R^c\mathbb{T}, P^c\mathbb{T})$ due to Connes and Kreimer.

1 Iterated integrals : smooth case

Let $t \mapsto \Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$ be a d -dimensional, compactly supported, *smooth* path. The purpose of this section is to give proper notations for iterated integrals of Γ and to introduce some tools which will pave the way for the regularization algorithm. Subsection 1.1 on tree iterated integrals is standard, see for instance [8]. We introduce *permutation graphs* and *Fourier normal ordering* for smooth paths in subsection 1.2. The *tree Chen property* – a generalization of the usual Chen property to tree iterated integrals – is recalled in subsection 1.3, in connection with the underlying Hopf algebraic structure. Finally, a variant of iterated integrals called *skeleton integrals* is introduced in subsection 1.4, together with a variant of the tree Chen property that we call *tree skeleton decomposition*.

1.1 From iterated integrals to trees

It was noted already long time ago [5] that iterated integrals could be encoded by trees, see also [20]. This remark has been exploited in connection with the construction of the rough path solution of partial, stochastic differential equations in [16]. The correspondence between trees and iterated integrals goes simply as follows.

Definition 1.1 *A decorated rooted tree (to be drawn growing up) is a finite tree with a distinguished vertex called root and edges oriented downwards, i.e. directed towards the root, such that every vertex wears a positive integer label called decoration.*

If \mathbb{T} is a decorated rooted tree, we let $V(\mathbb{T})$ be the set of its vertices (including the root), and $\ell : V(\mathbb{T}) \rightarrow \mathbb{N}$ be its decoration.

Definition 1.2 (tree partial ordering) Let \mathbb{T} be a decorated rooted tree.

- Letting $v, w \in V(\mathbb{T})$, we say that v connects directly to w , and write $v \rightarrow w$ or equivalently $w = v^-$, if (v, w) is an edge oriented downwards from v to w . Note that v^- exists and is unique except if v is the root.
- If $v_m \rightarrow v_{m-1} \rightarrow \dots \rightarrow v_1$, then we shall write $v_m \rightarrow v_1$, and say that v_m connects to v_1 . By definition, all vertices (except the root) connect to the root.
- Let $(v_1, \dots, v_{|V(\mathbb{T})|})$ be an ordering of $V(\mathbb{T})$. Assume that $(v_i \rightarrow v_j) \Rightarrow (i > j)$; in particular, v_1 is the root. Then we shall say that the ordering is compatible with the tree partial ordering defined by \rightarrow .

Definition 1.3 (tree integrals) (i) Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be a d -dimensional, compactly supported, smooth path, and \mathbb{T} a rooted tree decorated by $\ell : V(\mathbb{T}) \rightarrow \{1, \dots, d\}$. Then $I_{\mathbb{T}}(\Gamma) : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the iterated integral defined as

$$[I_{\mathbb{T}}(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_1}(\ell(v_1)) \int_s^{x_{v_2}^-} d\Gamma_{x_2}(\ell(v_2)) \dots \int_s^{x_{v_{|V(\mathbb{T})|}}^-} d\Gamma_{x_{v_{|V(\mathbb{T})|}}}(\ell(v_{|V(\mathbb{T})|})) \quad (1.1)$$

where $(v_1, \dots, v_{|V(\mathbb{T})|})$ is any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering.

In particular, if \mathbb{T} is a trunk tree with n vertices (see Fig. 1) – so that the tree ordering is total – we shall write

$$I_{\mathbb{T}}(\Gamma) = I_n^\ell(\Gamma), \quad (1.2)$$

where

$$[I_n^\ell(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)). \quad (1.3)$$

(ii) (multilinear extension) Assume μ is a compactly supported, signed Borel measure on $\mathbb{R}^{V(\mathbb{T})} := \{(x_v)_{v \in V(\mathbb{T})}, x_v \in \mathbb{R}\}$. Then

$$[I_{\mathbb{T}}(\mu)]_{ts} := \int_s^t \int_s^{x_{v_2}^-} \dots \int_s^{x_{v_{|V(\mathbb{T})|}}^-} \mu(dx_{v_1}, \dots, dx_{v_{|V(\mathbb{T})|}}). \quad (1.4)$$

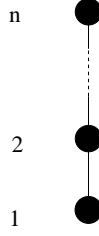


Figure 1: Trunk tree with set of vertices $\{n \rightarrow n-1 \rightarrow \dots \rightarrow 1\}$.

Clearly, the definition of $[I_{\mathbb{T}}(\Gamma)]_{ts}$ given in eq. (1.1) does not depend on the choice of the ordering $(v_1, \dots, v_{|V(\mathbb{T})|})$. For instance, consider $\mathbb{T} = \mathbb{T}_1^\sigma$ to be the first tree in Fig. 2. Then

$$\begin{aligned} [I_{\mathbb{T}}(\Gamma)]_{ts} &= \int_s^t d\Gamma_{x_1}(1) \left(\int_s^{x_1} d\Gamma_{x_2}(2) \int_s^{x_1} d\Gamma_{x_3}(3) \right) \\ &= \int_s^t d\Gamma_{x_1}(1) \left(\int_s^{x_1} d\Gamma_{x_2}(3) \int_s^{x_1} d\Gamma_{x_3}(2) \right). \end{aligned} \quad (1.5)$$

Note that the decoration of \mathbb{T} is required only for (i). In case of ambiguity, we shall also use the decoration-independent notation $I_{\mathbb{T}}(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(v)))$ instead of $I_{\mathbb{T}}(\Gamma)$.

The above correspondence extends by multilinearity to the algebra of decorated rooted trees defined by Connes and Kreimer [8], whose definition we now recall.

Definition 1.4 (algebra of decorated rooted trees) (i) Let \mathcal{T} be the set of decorated rooted trees.

(ii) Let \mathbf{H} be the free commutative algebra over \mathbb{R} generated by \mathcal{T} , with unit element denoted by e . If $\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_l$ are decorated rooted trees, then the product $\mathbb{T}_1 \dots \mathbb{T}_l$ is the forest with connected components $\mathbb{T}_1, \dots, \mathbb{T}_l$.

(iii) Let $\mathbb{T}' = \sum_{l=1}^L m_l \mathbb{T}_l \in \mathbf{H}$, where $m_l \in \mathbb{Z}$ and each $\mathbb{T}_l = \mathbb{T}_{l,1} \dots \mathbb{T}_{l,j_l}$ is a forest whose decorations have values in the set $\{1, \dots, d\}$. Then

$$[I_{\mathbb{T}'}(\Gamma)]_{ts} := \sum_{l=1}^L m_l [I_{\mathbb{T}_{l,1}}(\Gamma)]_{ts} \dots [I_{\mathbb{T}_{l,j_l}}(\Gamma)]_{ts}. \quad (1.6)$$

1.2 Permutation graphs and Fourier normal ordering for smooth paths

As explained briefly in the Introduction, and as we shall see in the next sections, an essential step in our regularization algorithm is to rewrite iterated integrals by permuting the order of integration. We shall prove the following lemma in this subsection:

Lemma 1.5 (permutation graphs) *To every trunk tree \mathbb{T}_n with n vertices and decoration ℓ , and every permutation $\sigma \in \Sigma_n$, is associated in a canonical way an element \mathbb{T}^σ of \mathbf{H} called permutation graph, such that:*

(i)

$$I_n^\ell(\Gamma) = I_{\mathbb{T}^\sigma}(\Gamma); \quad (1.7)$$

(ii)

$$\mathbb{T}^\sigma = \sum_{j=1}^{J_\sigma} g(\sigma, j) \mathbb{T}_j^\sigma \in \mathbf{H}, \quad (1.8)$$

where $g(\sigma, j) = \pm 1$ and each \mathbb{T}_j^σ , $j = 1, \dots, J_\sigma$ is a forest provided by construction with a total ordering compatible with its tree structure, image of the ordering $\{v_1 < \dots < v_n\}$ of the trunk tree \mathbb{T}_n by the permutation σ . The decoration of \mathbb{T}^σ is $\ell \circ \sigma$.

Proof. Let $\sigma \in \Sigma_n$. Applying Fubini's theorem yields

$$\begin{aligned} [I_n^\ell(\Gamma)]_{ts} &= \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)) \\ &= \int_{s_1}^{t_1} d\Gamma_{x_{\sigma(1)}}(\ell(\sigma(1))) \int_{s_2}^{t_2} d\Gamma_{x_{\sigma(2)}}(\ell(\sigma(2))) \dots \int_{s_n}^{t_n} d\Gamma_{x_{\sigma(n)}}(\ell(\sigma(n))), \end{aligned} \quad (1.9)$$

with $s_1 = s$, $t_1 = t$, and for some suitable choice of $s_j \in \{s\} \cup \{x_{\sigma(i)}, i < j\}$, $t_j \in \{t\} \cup \{x_{\sigma(i)}, i < j\}$ ($j \geq 2$). Now decompose $\int_{s_j}^{t_j} d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into

$$\left(\int_s^{t_j} - \int_s^{s_j} \right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$$

if $s_j \neq s$, $t_j \neq t$, and $\int_{s_j}^t d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into

$$\left(\int_s^t - \int_s^{s_j} \right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$$

if $s_j \neq s$. Then $I_n^\ell(\Gamma)$ has been rewritten as a sum of terms of the form

$$\pm \int_s^{\tau_1} d\Gamma_{x_1}(\ell(\sigma(1))) \int_s^{\tau_2} d\Gamma_{x_2}(\ell(\sigma(2))) \dots \int_s^{\tau_n} d\Gamma_{x_n}(\ell(\sigma(n))), \quad (1.10)$$

where $\tau_1 = t$ and $\tau_j \in \{t\} \cup \{x_i, i < j\}$, $j = 2, \dots, n$. Note the renaming of variables and vertices from eq. (1.9) to eq. (1.10). Encoding each of these expressions by the forest \mathbb{T} with set of vertices $V(\mathbb{T}) = \{1, \dots, n\}$, label function $\ell \circ \sigma$, roots $\{j = 1, \dots, n \mid \tau_j = t\}$, and oriented edges $\{(j, j^-) \mid j = 2, \dots, n, \tau_j = x_{j^-}\}$, yields

$$I_n^\ell(\Gamma) = I_{\mathbb{T}^\sigma}(\Gamma) \quad (1.11)$$

for some $\mathbb{T}^\sigma \in \mathbf{H}$ as in eq. (1.8). \square

Example 1.6 Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{t_2} d\Gamma_{x_2}(\ell(2)) \int_s^{t_3} d\Gamma_{x_3}(\ell(3)) = \\ - \int_s^t d\Gamma_{x_2}(\ell(2)) \int_s^{x_2} d\Gamma_{x_3}(\ell(3)) \int_s^{x_2} d\Gamma_{x_1}(\ell(1)) \\ + \int_s^t d\Gamma_{x_2}(\ell(2)) \int_s^{x_2} d\Gamma_{x_3}(\ell(3)) \cdot \int_s^t d\Gamma_{x_1}(\ell(1)) \end{aligned} \quad (1.12)$$

Hence $\mathbb{T}^\sigma = -\mathbb{T}_1^\sigma + \mathbb{T}_2^\sigma$ is the sum of a tree and of a forest with two components. See Fig. 2, where variables and vertices have been renamed according to the permutation σ .

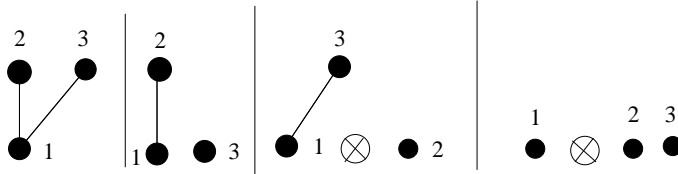


Figure 2: Example 1.6. From left to right: $\mathbb{T}_1^\sigma; \mathbb{T}_2^\sigma; \text{Roo}_{\{2\}}\mathbb{T}_1^\sigma \otimes \text{Lea}_{\{2\}}\mathbb{T}_1^\sigma; \text{Roo}_{\{2,3\}}\mathbb{T}_1^\sigma \otimes \text{Lea}_{\{2,3\}}\mathbb{T}_1^\sigma$

1.3 Tree Chen property and coproduct structure

The Chen property (ii), see Introduction, may be generalized to tree iterated integrals by using the coproduct structure of \mathbf{H} , as explained in [8]. It is an essential feature of our algorithm since it implies the possibility to reconstruct a rough path $\mathbf{\Gamma}$ from the quantities $t \mapsto \mathbf{\Gamma}_{ts_0}^n$ with *fixed* s_0 . This idea will be pursued further in the next subsection, where we shall introduce a variant of these iterated integrals with fixed s_0 called *skeleton integrals*.

Definition 1.7 (admissible cuts) (see [8], section 2)

1. Let \mathbb{T} be a tree, with set of vertices $V(\mathbb{T})$ and root denoted by 0. If $\mathbf{v} = (v_1, \dots, v_J)$, $J \geq 1$ is any totally disconnected subset of $V(\mathbb{T}) \setminus \{0\}$, i.e. $v_i \not\rightsquigarrow v_j$ for all $i, j = 1, \dots, J$, then we shall say that \mathbf{v} is an *admissible cut* of \mathbb{T} , and write $\mathbf{v} \models V(\mathbb{T})$. We let $Lea_{\mathbf{v}}\mathbb{T}$ (read: leaves of \mathbb{T}) be the sub-forest (or sub-tree if $J = 1$) obtained by keeping only the vertices above \mathbf{v} , i.e. $V(Lea_{\mathbf{v}}\mathbb{T}) = \mathbf{v} \cup \{w \in V(\mathbb{T}) : \exists j = 1, \dots, J, w \rightsquigarrow v_j\}$, and $Roo_{\mathbf{v}}\mathbb{T}$ (read: root part of \mathbb{T}) be the sub-tree obtained by keeping all other vertices.
2. Let $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ be a forest, together with its decomposition into trees. Then an *admissible cut* of \mathbb{T} is a disjoint union $\mathbf{v}_1 \cup \dots \cup \mathbf{v}_l$, $\mathbf{v}_i \subset \mathbb{T}_i$, where \mathbf{v}_i is either \emptyset , $\{0_i\}$ (root of \mathbb{T}_i) or an admissible cut of \mathbb{T}_i ; by convention, the two trivial cuts $\emptyset \cup \dots \cup \emptyset$ and $\{0_1\} \cup \dots \cup \{0_l\}$ are excluded. By definition, we let $Roo_{\mathbf{v}}\mathbb{T} = Roo_{\mathbf{v}_1}\mathbb{T}_1 \dots Roo_{\mathbf{v}_l}\mathbb{T}_l$, $Lea_{\mathbf{v}}\mathbb{T} = Lea_{\mathbf{v}_1}\mathbb{T}_1 \dots Lea_{\mathbf{v}_l}\mathbb{T}_l$ (if $\mathbf{v}_i = \emptyset$, resp. $\{0_i\}$, then $(Roo_{\mathbf{v}_i}\mathbb{T}_i, Lea_{\mathbf{v}_i}\mathbb{T}_i) := (\mathbb{T}_i, \emptyset)$, resp. $(\emptyset, \mathbb{T}_i)$).

See Fig. 3, 4 and 2. Defining the co-product operation

$$\Delta : \mathbf{H} \rightarrow \mathbf{H} \otimes \mathbf{H}, \quad \mathbb{T} \mapsto e \otimes \mathbb{T} + \mathbb{T} \otimes e + \sum_{\mathbf{v} \models V(\mathbb{T})} Roo_{\mathbf{v}}\mathbb{T} \otimes Lea_{\mathbf{v}}\mathbb{T} \quad (1.13)$$

where e stands for the unit element, yields a coalgebra structure on \mathbf{H} . One may also define an antipode S , which makes \mathbf{H} a Hopf algebra (see section 3 for more details).

We may now state the *tree Chen property*. Recall from the Introduction that $[\delta f]_{tus} := f_{ts} - f_{tu} - f_{us}$ if f is a function of two variables.

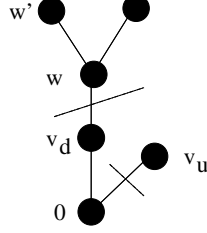


Figure 3: Admissible cut.

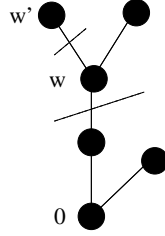


Figure 4: Non-admissible cut.

Proposition 1.8 (tree Chen property) (see [20] or [16])

Let \mathbb{T} be a forest, then

$$[\delta I_{\mathbb{T}}(\Gamma)]_{tus} = \sum_{\mathbf{v} \models V(\mathbb{T})} [I_{\text{Root } \mathbb{T}}(\Gamma)]_{tu} [I_{\text{Lea } \mathbf{v}}(\Gamma)]_{us}. \quad (1.14)$$

This proposition is illustrated in the discussion following Lemma 1.12 in the upcoming paragraph.

1.4 Skeleton integrals

We now introduce a variant of tree iterated integrals that we call *tree skeleton integrals*, or simply *skeleton integrals*. We explain after eq. (1.23) below the reason why we shall use skeleton integrals instead of usual iterated integrals as building stones for our construction.

Definition 1.9 (formal integral) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth, compactly supported function such that $\mathcal{F}f(0) = 0$. Then the formal integral $\int^t f$ of f is defined as

$$\int^t f := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \frac{e^{it\xi}}{i\xi} d\xi. \quad (1.15)$$

The condition $\mathcal{F}f(0) = 0$ prevents possible infra-red divergence when $\xi \rightarrow 0$. Note that

$$\int^t f - \int^s f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \left(\int_s^t e^{ix\xi} dx \right) d\xi = \int_s^t f(x) dx \quad (1.16)$$

by the Fourier inversion formula, so $\int^t f$ is an anti-derivative of f .

Formally one may write, as an equality of distributions:

$$\int^t e^{ix\xi} dx = \int_{-\infty}^t e^{ix\xi} dx = \frac{e^{it\xi}}{i\xi} \quad (1.17)$$

since $\int_{-\infty}^{+\infty} \frac{e^{ix\xi}}{i\xi} \phi(\xi) d\xi \rightarrow_{x \rightarrow \infty} 0$ for any test function ϕ such that $\phi(0) = 0$. Hence

$$\int^t f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi (\mathcal{F}f)(\xi) \int^t e^{ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \frac{e^{it\xi}}{i\xi} d\xi, \quad (1.18)$$

in coherence with eq. (1.15).

Definition 1.10 (skeleton integrals) (i) Let \mathbb{T} be a tree with decoration $\ell : \mathbb{T} \rightarrow \{1, \dots, d\}$. Let $(v_1, \dots, v_{|V(\mathbb{T})|})$ be any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering. Then the skeleton integral of Γ along \mathbb{T} is by definition

$$[\text{SkI}_{\mathbb{T}}(\Gamma)]_t := \int^{x_{v_1}} d\Gamma_{x_{v_1}}(\ell(v_1)) \int_{x_{v_1}}^{x_{v_2}} d\Gamma_{x_{v_2}}(\ell(v_2)) \dots \int_{x_{v_{|V(\mathbb{T})|}}}^{x_{v_{|V(\mathbb{T})|}}} d\Gamma_{x_{v_{|V(\mathbb{T})|}}}(\ell(v_{|V(\mathbb{T})|})). \quad (1.19)$$

(ii) (extension to forests) Let $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ be a forest, with its tree decomposition. Then one defines

$$[\text{SkI}_{\mathbb{T}}(\Gamma)]_t := \prod_{j=1}^l [\text{SkI}_{\mathbb{T}_j}(\Gamma)]_t. \quad (1.20)$$

(iii) (multilinear extension, see Definition 1.3) Assume \mathbb{T} is a subtree of $\tilde{\mathbb{T}}$, and μ a compactly supported, signed Borel measure on $\mathbb{R}^{\tilde{\mathbb{T}}} := \{(x_v)_{v \in V(\tilde{\mathbb{T}})}, x_v \in \mathbb{R}\}$. Then

$$[\text{SkI}_{\mathbb{T}}(\mu)]_t := \int^t \int_{x_{v_1}}^{x_{v_2}} \dots \int_{x_{v_{|V(\mathbb{T})|}}}^{x_{v_{|V(\mathbb{T})|}}} \mu(dx_{v_1}, \dots, dx_{v_{|V(\mathbb{T})|}}) \quad (1.21)$$

is a signed Borel measure on $\{(x'_v)_{v' \in V(\tilde{\mathbb{T}}) \setminus V(\mathbb{T})}, x_{v'} \in \mathbb{R}\}$.

Formally again, $[\text{SkI}_{\mathbb{T}}(\Gamma)]_t$ may be seen as $[I_{\mathbb{T}}(\Gamma)]_{t, \pm i\infty}$. Denote by $\hat{\mu}$ the partial Fourier transform of μ with respect to $(x_v)_{v \in V(\mathbb{T})}$, so that

$$\hat{\mu}((\xi_v)_{v \in V(\mathbb{T})}, (dx_{v'})_{v' \in V(\mathbb{T}) \setminus V(\mathbb{T})}) = (2\pi)^{-|V(\mathbb{T})|/2} \langle \mu, \left((x_v)_{v \in V(\mathbb{T})} \mapsto e^{-i \sum_{v \in V(\mathbb{T})} x_v \xi_v} \right) \rangle. \quad (1.22)$$

Then

$$[\text{SkI}_{\mathbb{T}}(\mu)]_t = (2\pi)^{-|V(\mathbb{T})|/2} \langle \hat{\mu}, \left[\text{SkI}_{\mathbb{T}} \left((x_v)_{v \in V(\mathbb{T})} \mapsto e^{i \sum_{v \in V(\mathbb{T})} x_v \xi_v} \right) \right]_t \rangle. \quad (1.23)$$

As explained in the previous subsection, tree skeleton integrals are straightforward generalizations of usual tree iterated integrals. They are very natural when computing in Fourier coordinates, because every successive integration brings about a new ξ -factor in the denominator, allowing easy Hölder estimates using Besov norms (see Appendix). On the contrary, $\int_0^t e^{ix\xi} dx = \frac{e^{it\xi}}{i\xi} - \frac{1}{i\xi}$ contains a constant term $-\frac{1}{i\xi}$ which does not improve when one integrates again.

It is the purpose of section 3 to show that a rough path $\mathbf{\Gamma}$ over an α -Hölder path Γ may be obtained from adequately *regularized* tree skeleton integrals, using the following *tree skeleton decomposition*, which is a variant of the *tree Chen property* recalled in Proposition 1.8 above.

Definition 1.11 (multiple cut) Let $\mathbf{v} \subset V(\mathbb{T})$, $\mathbf{v} \neq \emptyset$. If $w \in \mathbf{v}$, one calls $\text{Lev}(w) := 1 + |\{w' \in \mathbf{v}; w \twoheadrightarrow w'\}|$ the level of w . If $\mathbf{v} \models V(\mathbb{T})$ is an admissible cut, then $\text{Lev}(w) = 1$ for all $w \in \mathbf{v}$. Quite generally, letting $\text{Lev}(\mathbf{v}) = \max\{\text{Lev}(w); w \in \mathbf{v}\}$, one writes $\mathbf{v}_j := \{w \in \mathbf{v}; \text{Lev}(w) = j\}$ for $1 \leq j \leq \text{Lev}(\mathbf{v})$, and calls $(\mathbf{v}_j)_{j=1, \dots, \text{Lev}(\mathbf{v})}$ the level decomposition of \mathbf{v} considered as a multiple cut. One shall also write: $\mathbf{v}_1 \models \dots \models \mathbf{v}_{\text{Lev}(\mathbf{v})} \models V(\mathbb{T})$ since $\mathbf{v}_{\text{Lev}(\mathbf{v})} \models V(\mathbb{T})$ and each \mathbf{v}_j , $j = 1, \dots, \text{Lev}(\mathbf{v}) - 1$ is an admissible cut of $\text{Roo}_{\mathbf{v}_{j+1}}(\mathbb{T})$.

Lemma 1.12 (tree skeleton decomposition) Let \mathbb{T} be a tree. Then:

(i) (recursive version)

$$[I_{\mathbb{T}}(\Gamma)]_{tu} = [\delta \text{SkI}_{\mathbb{T}}(\Gamma)]_{tu} - \sum_{\mathbf{v} \models V(\mathbb{T})} [I_{\text{Roo}_{\mathbf{v}}\mathbb{T}}(\Gamma)]_{tu} \cdot [\text{SkI}_{\text{Lea}_{\mathbf{v}}\mathbb{T}}(\Gamma)]_u, \quad (1.24)$$

(ii) (non-recursive version)

$$\begin{aligned}
[I_{\mathbb{T}}(\Gamma)]_{tu} &= [\delta \text{SkI}_{\mathbb{T}}(\Gamma)]_{tu} + \sum_{l \geq 1} \sum_{\mathbf{v}_1 \models \dots \models \mathbf{v}_l \models V(\mathbb{T})} (-1)^{|\mathbf{v}_1| + \dots + |\mathbf{v}_l|} \\
&\quad [\delta \text{SkI}_{\text{Roo}_{\mathbf{v}_1}(\mathbb{T})}(\Gamma)]_{tu} \prod_{m=1}^{l-1} \left[\text{SkI}_{\text{Lea}_{\mathbf{v}_m} \circ \text{Roo}_{\mathbf{v}_{m+1}}(\mathbb{T})} \right]_u [\text{SkI}_{\text{Lea}_{\mathbf{v}_l}(\mathbb{T})}(\Gamma)]_u.
\end{aligned} \tag{1.25}$$

Proof. Same as for Proposition 1.8. Eq. (1.24) may formally be seen as a particular case of the Chen property (1.14) by setting $s = \pm i\infty$ (see previous subsection). The non-recursive version may be deduced from the recursive version in a straightforward way. \square

Let us illustrate these notions in a more pedestrian way for the reader who is not accustomed to tree integrals. Consider for an example the trunk tree \mathbb{T}_n with vertices $n \rightarrow n-1 \rightarrow \dots \rightarrow 1$ and decoration $\ell : \{1, \dots, n\} \rightarrow \{1, \dots, d\}$, and the associated iterated integral

$$[I_n^\ell(\Gamma)]_{ts} = [I_{\mathbb{T}_n}(\Gamma)]_{ts} = \int_s^t d\Gamma_{x_1}(\ell(1)) \dots \int_s^{x_{n-1}} d\Gamma_{x_n}(\ell(n)). \tag{1.26}$$

Cutting \mathbb{T}_n at some vertex $v \in \{2, \dots, n\}$ produces two trees, $\text{Roo}_v \mathbb{T}_n$ and $\text{Lea}_v \mathbb{T}_n$, with respective vertex subsets $\{1, \dots, v-1\}$ and $\{v, \dots, n\}$. Then the usual Chen property (ii) in the Introduction reads

$$[\delta I_{\mathbb{T}_n}(\Gamma)]_{tus} = \sum_{v \in V(\mathbb{T}_n) \setminus \{1\}} [I_{\text{Roo}_v \mathbb{T}_n}(\Gamma)]_{tu} [I_{\text{Lea}_v \mathbb{T}_n}(\Gamma)]_{us}. \tag{1.27}$$

On the other hand, rewrite $[I_{\mathbb{T}_n}(\Gamma)]_{tu}$ as the sum of the *increment term*, which is a skeleton integral,

$$\begin{aligned}
[\delta \text{SkI}_{\mathbb{T}_n}(\Gamma)]_{tu} &= \int^t d\Gamma_{x_1}(\ell(1)) \int^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int^{x_{n-1}} d\Gamma_{x_n}(\ell(n)) \\
&\quad - \int^u d\Gamma_{x_1}(\ell(1)) \int^{x_1} d\Gamma_{x_2}(\ell(2)) \dots \int^{x_{n-1}} d\Gamma_{x_n}(\ell(n))
\end{aligned} \tag{1.28}$$

and of the *boundary term*

$$\begin{aligned}
[I_{\mathbb{T}_n}(\Gamma)(\partial)]_{tu} &:= - \sum_{n_1 + n_2 = n} \int_u^t d\Gamma_{x_1}(\ell(1)) \dots \int_u^{x_{n_1-1}} d\Gamma_{x_{n_1}}(\ell(n_1)) \cdot \\
&\quad \cdot \int^u d\Gamma_{x_{n_1+1}}(\ell(n_1+1)) \int^{x_{n_1+1}} d\Gamma_{x_{n_1+2}}(\ell(n_1+2)) \dots \int^{x_{n-1}} d\Gamma_{x_n}(\ell(n)).
\end{aligned} \tag{1.29}$$

The above decomposition is fairly obvious for $n = 2$ and obtained by easy induction for general n . One has thus obtained the *recursive* skeleton decomposition property for trunk trees,

$$[I_{\mathbb{T}_n}(\Gamma)]_{tu} = [\delta \text{SkI}_{\mathbb{T}_n}(\Gamma)]_{tu} - \sum_{v \in V(\mathbb{T}_n) \setminus \{1\}} [I_{\text{Roo}_v \mathbb{T}_n}(\Gamma)]_{tu} \cdot [\text{SkI}_{\text{Lea}_v \mathbb{T}_n}(\Gamma)]_u. \quad (1.30)$$

The *non-recursive* version of the skeleton decomposition property is a straightforward consequence, and reads in this case

$$\begin{aligned} [I_{\mathbb{T}_n}(\Gamma)]_{tu} &= [\delta \text{SkI}_{\mathbb{T}_n}(\Gamma)]_{tu} + \sum_{l \geq 1} (-1)^l \times \\ &\times \sum_{j_1 < \dots < j_l} [\delta \text{SkI}_{\text{Roo}_{j_1}(\mathbb{T}_n)}(\Gamma)]_{tu} \prod_{m=1}^{l-1} [\text{SkI}_{\text{Lea}_{j_m} \circ \text{Roo}_{j_{m+1}}(\mathbb{T}_n)}(\Gamma)]_u [\text{SkI}_{\text{Lea}_{j_l}(\mathbb{T}_n)}(\Gamma)]_u, \end{aligned} \quad (1.31)$$

where $\text{Lea}_{j_m} \circ \text{Roo}_{j_{m+1}} \mathbb{T}_n$ is the piece of \mathbb{T}_n with subset of vertices ranging in $\{j_m, \dots, j_{m+1} - 1\}$.

2 Regularization : the Fourier normal ordering step by step.

We now come back to the original problem and assume Γ is a d -dimensional α -Hölder, compactly supported, *non-smooth* path. Then none of the previous definitions relative to iterated integrals make sense. However, one may rewrite these as diverging series such that every term is well-defined. This follows easily from the Besov decomposition given in the Appendix. Let us recall briefly, referring to the Appendix for details and notations, that Γ may be decomposed as $\sum_{k \in \mathbb{Z}} D(\phi_k) \Gamma$, where $(\phi_k)_{k \in \mathbb{Z}}$ is a dyadic partition of unity, and $D(\phi_k) \Gamma = \mathcal{F}^{-1}(\phi_k \cdot \mathcal{F} \Gamma)$. The Fourier transform \mathcal{F} has been introduced at the end of the Introduction. Since $\phi_k \cdot \mathcal{F} \Gamma$ is a compactly supported C^∞ function,

$$D(\phi_k) \Gamma : x \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi_k(\xi) (\mathcal{F} \Gamma)(\xi) e^{ix\xi} d\xi \quad (2.1)$$

is a C^∞ -function, and it makes perfectly sense to integrate the $D(\phi_k) \Gamma(i)$, $k \in \mathbb{Z}$, $1 \leq i \leq d$ against each other. We suggest the following definition, where $\mathbb{T} \in \mathcal{T}$ is a fixed tree. All \mathcal{P} -projections below extend to measures $\mu \in \text{Meas}(\mathbb{R}^{\mathbb{T}})$, where $\mathbb{R}^{\mathbb{T}} := \{(x_v)_{v \in V(\mathbb{T})}, x_v \in \mathbb{R}\}$.

Definition 2.1 (\mathcal{P} -projections) (i) Let, for $\mathbf{k} \in \mathbb{Z}^{\mathbb{T}} := \{(k_v)_{v \in V(\mathbb{T})}, k_v \in \mathbb{Z}\}$,

$$\mathcal{P}^{\{\mathbf{k}\}}(\Gamma) := \otimes_{v \in V(\mathbb{T})} D(\phi_{k_v}) \Gamma(\ell(v)), \quad (2.2)$$

(ii) Similarly, let $U \subset \mathbb{Z}^{\mathbb{T}}$. Then

$$\mathcal{P}^U(\Gamma) := \sum_{\mathbf{k}=(k_v)_{v \in V(\mathbb{T})} \in U} \mathcal{P}^{\{\mathbf{k}\}}(\Gamma). \quad (2.3)$$

(iii) Let in particular $\mathcal{P}^{+, \mathbb{T}}$ be the \mathcal{P} -projection associated to the subset

$$U = \mathbb{Z}_+^{\mathbb{T}} := \{(k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}} \mid (v \rhd w) \Rightarrow |k_v| \geq |k_w|\}. \quad (2.4)$$

If $\mathbb{T} = \mathbb{T}_n$ is the trunk tree with n vertices $\{n \rightarrow \dots \rightarrow 1\}$ and decoration $\ell : j \mapsto j$, $j = 1, \dots, n$, see Fig. 1, we shall simply write \mathcal{P}^+ instead of $\mathcal{P}^{+, \mathbb{T}_n}$. More generally, if a tree \mathbb{T} is equipped with a partial or total ordering $>$ compatible with its tree ordering, we let $\mathcal{P}^+ := \mathcal{P}^{U>}$ with $U_{>} := \{(k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}} \mid (v > w) \Rightarrow |k_v| \geq |k_w|\}$.

(iv) Using the Fourier multipliers $D(\tilde{\phi}_{k_v})$ instead of $D(\phi_{k_v})$, see Definition 5.3, define similarly

$$\tilde{\mathcal{P}}^{\{\mathbf{k}\}} := \frac{1}{|\Sigma_{\mathbf{k}}|} \otimes_{v \in V(\mathbb{T})} D(\tilde{\phi}_{k_v}) \Gamma(\ell(v)), \quad (2.5)$$

where $\Sigma_{\mathbf{k}} \subset \Sigma_n$ is the subset of permutations τ such that $|k_{\tau(j)}| = |k_j|$ for every $j = 1, \dots, n$, and

$$\tilde{\mathcal{P}}^+ := \sum_{\mathbf{k}=(k_v)_{v \in V(\mathbb{T})} \in U_{>}} \tilde{\mathcal{P}}^{\{\mathbf{k}\}}(\Gamma). \quad (2.6)$$

Remark. By construction, $\mathcal{P}^+ \tilde{\mathcal{P}}^+ = \tilde{\mathcal{P}}^+$ if \mathcal{P}^+ , $\tilde{\mathcal{P}}^+$ are associated to a total ordering compatible with the tree ordering of \mathbb{T} .

Note that \mathcal{P}^U may be considered as a linear operator $\mathcal{P}^U : (B_{\infty, \infty}^{\alpha})^{\otimes \mathbb{T}} \rightarrow (B_{\infty, \infty}^{\alpha})^{\otimes \mathbb{T}}$, where $(B_{\infty, \infty}^{\alpha})^{\otimes \mathbb{T}}$ stands for the vector space generated by the monomials $\otimes_{v \in V(\mathbb{T})} f_v$, $f_v \in B_{\infty, \infty}^{\alpha}$. It is actually a bounded linear operator, as recalled in the Appendix, see Proposition 5.8 and remarks after Proposition 5.2.

We may now proceed to explain our *regularization algorithm*.

- *Step 1 (choice of regularization scheme).* Choose for each tree $\mathbb{T} \in \mathcal{T}$ a subset $\mathbb{Z}_{reg}^{\mathbb{T}} \subset \mathbb{Z}_+^{\mathbb{T}}$ such that the series $\sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}} [\text{SkI}_{\mathbb{T}}(\mathcal{P}^{\{\mathbf{k}\}}(\Gamma))]_t$ converges absolutely for *any* α -Hölder path Γ . By assumption $\mathbb{Z}_{reg}^{\mathbb{T}} = \mathbb{Z}$ if $|V(\mathbb{T})| = 1$.
- *Step 2.* Let \mathbb{T} be a forest equipped with a partial or total ordering compatible with its tree ordering, and $\tilde{\mathcal{P}}^+$ the corresponding projection operator. For $\mathbf{k} \in \mathbb{Z}_+^{\mathbb{T}}$, we let the *projected regularized skeleton integral* be the quantity

$$[\mathcal{R}^{\{\mathbf{k}\}} \text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma)]_t = \mathbf{1}_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}} \cdot [\text{SkI}_{\mathbb{T}}(\mathcal{P}^{\{\mathbf{k}\}} \tilde{\mathcal{P}}^+ \Gamma)]_t. \quad (2.7)$$

- *Step 3 (regularized projected tree integral).* For $\mathbf{k} \in \mathbb{Z}_+^{\mathbb{T}}$, let $[\mathcal{R}^{\{\mathbf{k}\}} I_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma)]_{ts}$ be constructed out of projected regularized skeleton integrals in the following recursive way, as in Lemma 1.12:

$$[\mathcal{R}^{\{\mathbf{k}\}} I_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma)]_{ts} := [\delta \mathcal{R}^{\{\mathbf{k}\}} \text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma)]_{ts} - \sum_{v \models V(\mathbb{T})} [\mathcal{R}^{\{Roo_v(\mathbf{k})\}} I_{Roo_v(\mathbb{T})}(\tilde{\mathcal{P}}^+ \Gamma)]_{ts} [\mathcal{R}^{\{Lea_v(\mathbf{k})\}} \text{SkI}_{Lea_v \mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma)]_s, \quad (2.8)$$

where $Roo_v(\mathbf{k}) = (k_w)_{w \in Roo_v(\mathbb{T})} \in \mathbb{Z}^{Roo_v(\mathbb{T})}$, and $Lea_v(\mathbf{k}) = (k_w)_{w \in Lea_v(\mathbb{T})} \in \mathbb{Z}^{Lea_v(\mathbb{T})}$.

- *Step 4 (generalization to forests).* The generalization is straightforward. Namely, if $\mathbb{T} = \mathbb{T}_1 \dots \mathbb{T}_l$ is a forest, and $\mathbf{k} = (\mathbf{k}_1, \dots, \mathbf{k}_l) \in \mathbb{Z}_+^{\mathbb{T}_1} \times \dots \times \mathbb{Z}_+^{\mathbb{T}_l}$, we let

$$\mathcal{R}^{\{\mathbf{k}\}} \text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma) := \prod_{j=1}^l \mathcal{R}^{\{\mathbf{k}_j\}} \text{SkI}_{\mathbb{T}_j}(\tilde{\mathcal{P}}^+ \Gamma) \quad (2.9)$$

and similarly

$$\mathcal{R}^{\{\mathbf{k}\}} I_{\mathbb{T}}(\tilde{\mathcal{P}}^+ \Gamma) := \prod_{j=1}^l \mathcal{R}^{\{\mathbf{k}_j\}} I_{\mathbb{T}_j}(\tilde{\mathcal{P}}^+ \Gamma). \quad (2.10)$$

Consider a partial or total ordering $>$ on \mathbb{T} and denote by $\tilde{\mathcal{P}}^+$ the corresponding projection operator. By summing over all indices

$\mathbf{k} \in U_{>}$, one gets the following quantities,

$$\mathcal{R}\text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma) := \sum_{\mathbf{k} \in U_{>}} \mathcal{R}^{\{\mathbf{k}\}} \text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma) \quad (2.11)$$

(see Definition 2.1), and similarly

$$\mathcal{R}I_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma) := \sum_{\mathbf{k} \in U_{>}} \mathcal{R}^{\{\mathbf{k}\}} I_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma). \quad (2.12)$$

Observe in particular, using eq. (2.8), and summing over indices \mathbf{k} , that $\mathcal{R}I_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma)$ decomposes naturally into the sum of an *increment term*, which is a regularized skeleton integral, and of a *boundary term* denoted by the symbol ∂ , namely,

$$\left[\delta \mathcal{R}\text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma) \right]_{ts} + \left[\mathcal{R}I_{\mathbb{T}}(\tilde{\mathcal{P}}^+\Gamma)(\partial) \right]_{ts}. \quad (2.13)$$

This decomposition is a generalization of that obtained in subsection 1.4, see eq. (1.28) and (1.29). Observe also that we *have not defined* $\mathcal{R}\text{SkI}_{\mathbb{T}}(\Gamma)$, nor $\mathcal{R}I_{\mathbb{T}}(\Gamma)$; the regularized integration operators $\mathcal{R}I_{\mathbb{T}}$, $\mathcal{R}\text{SkI}_{\mathbb{T}}$ only act on *Fourier normal ordered projections of paths* $\tilde{\mathcal{P}}^+\Gamma$.

- *Final step (Fourier normal ordering).* Let \mathbb{T}_n be a trunk tree with n vertices decorated by ℓ , and, for each $\sigma \in \Sigma_n$, $\mathbb{T}^\sigma = \sum_{j=1}^{J_\sigma} g(\sigma, j) \mathbb{T}_j^\sigma$ be the corresponding permutation graph, as in Lemma 1.5. Each forest \mathbb{T}^σ comes with a total ordering compatible with its tree ordering, which defines a projection operator $\tilde{\mathcal{P}}^+$; we write for short $\tilde{\mathcal{P}}^\sigma \Gamma$ instead of $\tilde{\mathcal{P}}^+(\otimes_{m=1}^n \Gamma(\ell(\sigma(m))))$. Then we let

$$\begin{aligned} [\mathcal{R}\Gamma^n(\ell(1), \dots, \ell(n))]_{ts} &:= \sum_{\sigma \in \Sigma_n} \sum_{j=1}^{J_\sigma} g(\sigma, j) \mathcal{R}I_{\mathbb{T}_j^\sigma}(\tilde{\mathcal{P}}^\sigma \Gamma) \\ &= \sum_{\sigma \in \Sigma_n} \left(\sum_{\mathbf{k}=(k_1, \dots, k_n) \in \mathbb{Z}^n ; |k_{\sigma(1)}| \leq \dots \leq |k_{\sigma(n)}|} \sum_{j=1}^{J_\sigma} g(\sigma, j) [\mathcal{R}^{\{\mathbf{k} \circ \sigma\}} I_{\mathbb{T}_j^\sigma}(\tilde{\mathcal{P}}^\sigma \Gamma)]_{ts} \right). \end{aligned} \quad (2.14)$$

We shall prove in the next section that $\mathcal{R}\Gamma$ satisfies the Chen (ii) and shuffle (iii) properties of the Introduction. The Hölder property (i) will be proved in section 4 for an adequate choice of subdomains $\mathbb{Z}_{reg}^\mathbb{T}$, $\mathbb{T} \in \mathcal{T}$ satisfying in particular the property required in Step 1.

Some essential comments are in order.

1. Assume that Γ is smooth, and *do not regularize*, i.e., choose $\mathbb{Z}_{reg}^T = \mathbb{Z}_+^T$. Then eq. (2.8) is a recursive definition of the *non-regularized* projected integral $[I_{\mathbb{T}}(\mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^+\Gamma)]_{ts}$, as follows from the tree skeleton decomposition property, see Lemma 1.12. Hence the right-hand side of formula (2.14) reads simply

$$\sum_{\sigma \in \Sigma_n} \sum_{\mathbf{k}=(k_1, \dots, k_n) \in \mathbb{Z}^n ; |k_{\sigma(1)}| \leq \dots \leq |k_{\sigma(n)}|} \sum_{j=1}^{J_\sigma} g(\sigma, j) [I_{\mathbb{T}_j^\sigma}(\mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^\sigma \Gamma)]_{ts}. \quad (2.15)$$

But this quantity is the usual iterated integral or canonical lift of Γ , $[\mathbf{\Gamma}^{cano,n}(\ell(1), \dots, \ell(n))]_{ts}$, since

$$\sum_{j=1}^{J_\sigma} g(\sigma, j) [I_{\mathbb{T}_j^\sigma}(\mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^\sigma \Gamma)]_{ts} = [I_{\mathbb{T}^\sigma}(\mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^\sigma \Gamma)]_{ts} = [I_n^\ell(\mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^\sigma \Gamma)]_{ts} \quad (2.16)$$

by Lemma 1.5, and

$$\begin{aligned} \sum_{\sigma \in \Sigma_n} \sum_{\mathbf{k}=(k_1, \dots, k_n) \in \mathbb{Z}^n ; |k_{\sigma(1)}| \leq \dots \leq |k_{\sigma(n)}|} \mathcal{P}^{\{\mathbf{k}\}}\tilde{\mathcal{P}}^\sigma(\Gamma) &= \sum_{\sigma \in \Sigma_n} \mathcal{P}^+\tilde{\mathcal{P}}^+(\otimes_{m=1}^n \Gamma(\ell(\sigma(m)))) \\ &= \sum_{\sigma \in \Sigma_n} \tilde{\mathcal{P}}^+(\otimes_{m=1}^n \Gamma(\ell(\sigma(m)))) = \Gamma, \end{aligned} \quad (2.17)$$

see Remark after Definition 2.1.

2. Iterated integrals of order 1, $[\mathcal{R}\Gamma^1(i)]_{ts}$, $1 \leq i \leq d$, are *not regularized*, namely, $[\mathcal{R}\Gamma^1(i)]_{ts} = [\mathbf{\Gamma}^1(i)]_{ts} = \Gamma_t(i) - \Gamma_s(i)$, because of the assumption in Step 1 which states that $\mathbb{Z}_{reg}^T = \mathbb{Z}$ if $|V(\mathbb{T})| = 1$. Hence $\mathcal{R}\Gamma$ is a rough path over Γ .

3. We propose a reformulation of this algorithm in a Hopf algebraic language in Lemma 3.5 below. An equivalent algorithm is given in Definition 3.7. The abstract algebraic language of section 3 turns out to be very appropriate to prove the Chen and shuffle properties.

3 Proof of the geometric and multiplicative properties

Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be an α -Hölder path. This section is dedicated to the proof of

Theorem 3.1 *Choose for each tree \mathbb{T} a subset $\mathbb{Z}_{reg}^{\mathbb{T}} \subset \mathbb{Z}^{\mathbb{T}}$ such that the condition of Step 1 of the construction in section 2 is satisfied, i.e. such that the regularized rough path $\mathcal{R}\Gamma$ defined in section 2 is well-defined. Then $\mathcal{R}\Gamma$ satisfies the Chen (ii) and shuffle (iii) properties of the Introduction.*

This theorem is in fact a consequence of the following very general construction, whose essence is really algebraic. Two Hopf algebras are involved in it: the *Hopf algebra of decorated rooted trees* \mathbf{H} , and the *shuffle algebra* \mathbf{Sh} . As we shall presently see, the first one is related to the *Chen property*, while the second one is related to the *shuffle property*. The first paragraph below is devoted to an elementary presentation of these Hopf algebras in connection with the Chen/shuffle property. Theorem 3.1 is proved in the second paragraph.

3.1 Hopf algebras and the Chen and shuffle properties

1. Let us first consider the Hopf algebra of decorated rooted trees, \mathbf{H} . Recall the definition of the coproduct on \mathbf{H} ,

$$\Delta(\mathbb{T}) = e \otimes \mathbb{T} + \mathbb{T} \otimes e + \sum_{v \models V(\mathbb{T})} Roo_v \mathbb{T} \otimes Lea_v \mathbb{T}. \quad (3.1)$$

The usual convention [8, 9] is to write c (cut) for v , $R^c(\mathbb{T})$ (*root part*) for $Roo_v \mathbb{T}$, $P^c(\mathbb{T})$ for $Lea_v \mathbb{T}$ (*leaves*), and to reverse the order of the factors in the tensor product.

The *convolution* of two linear forms f, g on \mathbf{H} writes

$$(f * g)(\mathbb{T}) = f(\mathbb{T})g(e) + f(e)g(\mathbb{T}) + \sum_{v \models V(\mathbb{T})} f(Roo_v \mathbb{T})g(Lea_v \mathbb{T}), \quad \mathbb{T} \in \mathbf{H}. \quad (3.2)$$

This notion is particularly interesting for *characters*. A character of \mathbf{H} is a linear map such that $\chi(\mathbb{T}_1 \cdot \mathbb{T}_2) = \chi(\mathbb{T}_1) \cdot \chi(\mathbb{T}_2)$. If χ_1, χ_2 are two characters of \mathbf{H} , then $\chi_1 * \chi_2$ is also a character of \mathbf{H} .

The *tree Chen property*, see Proposition 1.8, may then be stated as follows. Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be a smooth path, and

$$\mathbf{H}^d := \{\mathbb{T} \in \mathbf{H}; \ell : V(\mathbb{T}) \rightarrow \{1, \dots, d\}\} \quad (3.3)$$

be the subspace of \mathbf{H} generated by forests with decoration valued in $\{1, \dots, d\}$. Now, define $I_\Gamma^{ts} : \mathbf{H}^d \rightarrow \mathbb{R}$ to be the following character of \mathbf{H} (see Definition 1.3)

$$I_\Gamma^{ts}(\mathbb{T}) = [I_\Gamma(\Gamma)]_{ts}. \quad (3.4)$$

Then (as remarked in [20])

$$I_{\Gamma}^{ts} = I_{\Gamma}^{tu} * I_{\Gamma}^{us}. \quad (3.5)$$

Generalizing this property to the multilinear setting, one may also write

$$I_{\mu}^{ts}(\mathbb{T}) = (I^{tu} * I^{us})_{\mu}(\mathbb{T}) := I_{\mu}^{tu}(\mathbb{T}) + I_{\mu}^{us}(\mathbb{T}) + \sum_{\mathbf{v} \models V(\mathbb{T})} I_{Roo_{\mathbf{v}}(\mu)}^{tu}(Roo_{\mathbf{v}}(\mathbb{T})) I_{Lea_{\mathbf{v}}(\mu)}^{us}(Lea_{\mathbf{v}}(\mathbb{T})) \quad (3.6)$$

for a tensor measure $\mu = \otimes_{v \in V(\mathbb{T})} \mu_v$, where $Roo_{\mathbf{v}}(\mu) := \otimes_{v \in V(Roo_{\mathbf{v}}(\mathbb{T}))} \mu_v$, $Lea_{\mathbf{v}}(\mu) := \otimes_{v \in V(Lea_{\mathbf{v}}(\mathbb{T}))} \mu_v$, and

$$I_{\mu}^{ts}(\mathbb{T}) := (I^{tu} * I^{us})_{\mu}(\mathbb{T}) := \sum_{\mathbf{k}} (I^{tu} * I^{us})_{\mu_{\mathbf{k}}}(\mathbb{T}) \quad (3.7)$$

for a more general measure $\mu := \sum_{\mathbf{k}} \mu_{\mathbf{k}}$, where each $\mu_{\mathbf{k}}$ is a tensor measure. Later on we shall use these formulas for $\mu_{\mathbf{k}} = \mathbf{1}_{\mathbf{k} \in \mathbb{Z}_{+}^{\mathbb{T}}} d\mathcal{P}^{\{\mathbf{k}\}}(\Gamma)$ or $\mathbf{1}_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}} d\mathcal{P}^{\{\mathbf{k}\}}(\Gamma)$.

As for the *antipode* S , it is the multiplicative morphism $S : \mathbf{H} \rightarrow \mathbf{H}$ defined inductively on tree generators \mathbb{T} by (see [8], p. 219)

$$S(e) = e; \quad S(\mathbb{T}) = -\mathbb{T} - \sum_{\mathbf{v} \models V(\mathbb{T})} Roo_{\mathbf{v}} \mathbb{T} \cdot S(Lea_{\mathbf{v}} \mathbb{T}). \quad (3.8)$$

Applying iteratively the second relation yields an expression of $S(\mathbb{T})$ in terms of multiple cuts of \mathbb{T} obtained by 'chopping' it [8], see Definition 1.11, namely,

$$S(\mathbb{T}) = -\mathbb{T} - \sum_{l \geq 1} \sum_{\mathbf{v}_1 \models \dots \models \mathbf{v}_l \models V(\mathbb{T})} (-1)^{|\mathbf{v}_1| + \dots + |\mathbf{v}_l|} Roo_{\mathbf{v}_1}(\mathbb{T}) \left\{ \prod_{m=1}^{l-1} Lea_{\mathbf{v}_m} \circ Roo_{\mathbf{v}_{m+1}}(\mathbb{T}) \right\} Lea_{\mathbf{v}_l}(\mathbb{T}). \quad (3.9)$$

Let χ_1, χ_2 be two characters of \mathbf{H} . Recall that $\chi_2 \circ S$ is the *convolution inverse* of χ_2 , namely, $\chi_2 \circ S$ is a character and $\chi_2 * (\chi_2 \circ S) = \bar{e}$, where

\bar{e} is the counit of \mathbf{H} , defined on generators by $\bar{e}(e) = 1$ and $\bar{e}(\mathbb{T}) = 0$ if \mathbb{T} is a forest. Now eq. (3.2) and (3.9) yield

$$\begin{aligned}
\chi_1 * (\chi_2 \circ S)(\mathbb{T}) &= \chi_1(\mathbb{T}) + \chi_2 \circ S(\mathbb{T}) + \sum_{\mathbf{v} \models V(\mathbb{T})} \chi_1(Roo_{\mathbf{v}}(\mathbb{T})) \chi_2 \circ S(Lea_{\mathbf{v}}(\mathbb{T})) \\
&= (\chi_1 - \chi_2)(\mathbb{T}) + \sum_{\mathbf{v} \models V(\mathbb{T})} (\chi_1 - \chi_2)(Roo_{\mathbf{v}}(\mathbb{T})) \chi_2 \circ S(Lea_{\mathbf{v}}(\mathbb{T})) \\
&= (\chi_1 - \chi_2)(\mathbb{T}) + \sum_{l \geq 1} (-1)^{|\mathbf{v}_1| + \dots + |\mathbf{v}_l|} \sum_{\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_l)} (\chi_1 - \chi_2)(Roo_{\mathbf{v}_1}(\mathbb{T})) \times \\
&\quad \times \left[\prod_{m=1}^{l-1} \chi_2(Lea_{\mathbf{v}_m} \circ Roo_{\mathbf{v}_{m+1}}(\mathbb{T})) \right] \chi_2(Lea_{\mathbf{v}_l}(\mathbb{T}))
\end{aligned} \tag{3.10}$$

where $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_l)$ is a multiple cut of \mathbb{T} as in eq. (3.9).

In particular, let $\text{SkI}_{\Gamma}^t : \mathbf{H} \rightarrow \mathbb{R}$ be the character defined by (see Definition 1.10)

$$\text{SkI}_{\Gamma}^t(\mathbb{T}) = [\text{SkI}_{\Gamma}(\Gamma)]_t. \tag{3.11}$$

Then the tree skeleton decomposition, see Lemma 1.12, reads simply

$$I_{\Gamma}^{tu} = \text{SkI}_{\Gamma}^t * (\text{SkI}_{\Gamma}^u \circ S). \tag{3.12}$$

2. The *shuffle algebra* over the index set \mathbb{N} [24] may be defined as follows. The algebra \mathbf{Sh} is generated as a vector space over \mathbb{R} by the identity e and by the trunk trees $(\mathbb{T}_n)_{n \geq 1}$ with vertex set $V(\mathbb{T}_n) = \{v_1 < \dots < v_n\}$, provided with an \mathbb{N} -valued decoration ℓ . Let $\mathbb{T}_n, \mathbb{T}'_{n'}$ be trunk trees with n , resp. n' vertices. The *shuffle product* of \mathbb{T}_n and $\mathbb{T}'_{n'}$ is the formal sum

$$\mathbb{T}_n \circ \mathbb{T}'_{n'} = \sum_{\varepsilon \in Sh((V(\mathbb{T}_n), V(\mathbb{T}'_{n'})))} \varepsilon(\mathbb{T}_n^{\mathbb{T}'_{n'}}), \tag{3.13}$$

where $\mathbb{T}_n^{\mathbb{T}'_{n'}}$ is the trunk tree with $n + n'$ vertices obtained by putting $\mathbb{T}'_{n'}$ on top of \mathbb{T}_n , and the shuffle ε permutes the decorations of $\mathbb{T}_n, \mathbb{T}'_{n'}$ as in property (iii) discussed in the Introduction.

Let \mathbf{Sh}^d be the subspace of \mathbf{Sh} generated by trunk trees with decoration valued in $\{1, \dots, d\}$. Then the *shuffle property* for iterated integrals reads

$$I_{\Gamma}^{ts}(\mathbb{T}_n) I_{\Gamma}^{ts}(\mathbb{T}'_{n'}) = I_{\Gamma}^{ts}(\mathbb{T}_n \circ \mathbb{T}'_{n'}), \quad \mathbb{T}_n, \mathbb{T}'_{n'} \in \mathbf{Sh}^d. \tag{3.14}$$

In other words, it may be stated by saying that $I_\Gamma^{ts} : \mathbb{T}_n \rightarrow [I_{\mathbb{T}_n}(\Gamma)]_{ts}$ is a character of **Sh**. Similarly, skeleton integrals $\text{SkI}_\Gamma^t : \mathbb{T}_n \rightarrow [\text{SkI}_\mathbb{T}(\Gamma)]_t$ also define characters of **Sh**.

The shuffle algebra **Sh** is made into a Hopf algebra by re-using the same coproduct $\Delta : \mathbb{T} \rightarrow \mathbb{T} \otimes e + e \otimes \mathbb{T} + \sum_{v \models V(\mathbb{T})} \text{Roo}_v \mathbb{T} \otimes \text{Lea}_v \mathbb{T}$ as for **H**, and defining the antipode \bar{S} as $\bar{S}(\mathbb{T}_n) = (-1)^n \bar{\mathbb{T}}_n$, where $\bar{\mathbb{T}}_n$ is obtained from \mathbb{T}_n by reversing the ordering of the vertices, $\ell_{\bar{\mathbb{T}}_n}(v_j) = \ell_{\mathbb{T}_n}(v_{n+1-j})$.

The convolution of linear forms or characters f, g on **Sh** is given by the same formula as for **H**.

Proposition 3.1 [24]

The linear morphism $\Pi : \mathbf{H} \rightarrow \mathbf{Sh}$ defined by $\Pi(\mathbb{T}) = \sum_j \mathbb{T}_j$, where \mathbb{T}_j ranges over all trunk trees $\{v_1 < \dots < v_{|V(\mathbb{T})|}\}$ such that the corresponding total ordering of vertices of \mathbb{T} is compatible with its tree partial ordering, is a Hopf algebra map.

Π is actually *onto*. In other words, it is a *structure-preserving projection*, with the canonical identification of **Sh** as a subspace of \mathbb{T} . Note that $[I_\mathbb{T}(\Gamma)]_{ts} = [\text{SkI}_\mathbb{T}(\Gamma)]_{ts} = 0$ if $\mathbb{T} \in \text{Ker}(\Pi)$ and Γ is an arbitrary smooth path, which is a straightforward generalization of the shuffle property; one may call this the *tree shuffle property*.

Corollary 3.2 *Let $\bar{\chi}$ be a character of **Sh**. Then $\chi := \bar{\chi} \circ \Pi$ is a character of **H**. If $\mathbb{T} \in \mathbf{Sh}$, then $\chi \circ S(\mathbb{T}) = \bar{\chi} \circ \bar{S}(\mathbb{T})$.*

3.2 Proof of the Chen and shuffle properties

We shall now prove Theorem 3.1. In the next pages, $\text{Meas}(\mathbb{R}^n)$ stands for the space of compactly supported, signed Borel measures on \mathbb{R}^n . Let us explain the strategy of the proof. We give a general method to construct families of characters of the shuffle algebra, $\bar{\chi}_\Gamma^t$, depending on a path Γ , see Lemma 3.6; these quantities satisfy the shuffle property by eq. (3.14). Then $\bar{\chi}_\Gamma^t * (\bar{\chi}^s \circ \bar{S})$ is immediately seen to define a rough path satisfying both the Chen and shuffle properties, see Definition 3.7. For a particular choice of the characters $\bar{\chi}_\Gamma^t$ related to the regularized skeleton integrals defined in section 2, the rough path of Definition 3.7 is shown to coincide with the regularized rough path $\mathcal{R}\Gamma$ of section 2, see Lemma 3.8. In order to prove

this last lemma, one needs a Hopf algebraic reformulation of the Fourier normal ordering algorithm leading to $\mathcal{R}\Gamma$, see Lemma 3.5.

Lemma 3.3 (measure splitting) *Let $\mu \in \text{Meas}(\mathbb{R}^n)$. Then*

$$\mu = \sum_{\sigma \in \Sigma_n} \mu^\sigma \circ \sigma, \quad (3.15)$$

where $\mu^\sigma \in \tilde{\mathcal{P}}^+ \text{Meas}(\mathbb{R}^n)$ is defined by

$$\mu^\sigma := \sum_{\mathbf{k}=(k_1, \dots, k_n) \in \mathbb{Z}^n; |k_{\sigma(1)}| \leq \dots \leq |k_{\sigma(n)}|} (\tilde{\mathcal{P}}^{\{\mathbf{k}\}} \mu) \circ \sigma \quad (3.16)$$

as in eq. (2.14).

Proof. See eq. (2.17). \square

Definition 3.4 (i) Let $\mathcal{F}_{n,n}^+ \subset \mathbf{H}$ ($n \geq 1$) be the set of all forests \mathbb{T} with n vertices and one-to-one decoration $\ell : V(\mathbb{T}) \rightarrow \{1, \dots, n\}$ valued in the set $\{1, \dots, n\}$, such that $(v \rightarrow w) \Rightarrow \ell(v) \geq \ell(w)$, and $\mathbf{H}_{n,n}^+ \subset \mathbf{H}$ the vector space generated by $\mathcal{F}_{n,n}^+$.

(ii) If $\mathbb{T} \in \mathcal{F}_{n,n}^+$, let $\tilde{\mathcal{P}}^{+, \mathbb{T}} \text{Meas}(\mathbb{R}^n)$ denote the subspace $\{\tilde{\mathcal{P}}^{+, \mathbb{T}} \mu; \mu \in \text{Meas}(\mathbb{R}^n)\}$, see section 2 for a definition of the projection operator $\tilde{\mathcal{P}}^{+, \mathbb{T}}$.

(iii) Let $\phi_{\mathbb{T}}^t : \tilde{\mathcal{P}}^{+, \mathbb{T}} \text{Meas}(\mathbb{R}^n) \rightarrow \mathbb{R}, \mu \mapsto \phi_{\mathbb{T}}^t(\mu)$, also written $\phi_{\mathbb{T}}^t(\mathbb{T})$ ($t \in \mathbb{R}, \mathbb{T} \in \mathcal{F}_{n,n}^+$) be a family of linear forms such that, if $(\mathbb{T}_i, \mu_i) \in \mathcal{F}_{n_i, n_i}^+ \times \tilde{\mathcal{P}}^{+, \mathbb{T}_i} \text{Meas}(\mathbb{R}^{n_i})$, $i = 1, 2$, the following \mathbf{H} -multiplicative property holds,

$$\phi_{\mu_1}^t(\mathbb{T}_1) \phi_{\mu_2}^t(\mathbb{T}_2) = \phi_{\mu_1 \otimes \mu_2}^t(\mathbb{T}_1 \wedge \mathbb{T}_2), \quad (3.17)$$

where $\mathbb{T}_1 \wedge \mathbb{T}_2 \in \mathcal{F}_{n_1+n_2, n_1+n_2}^+$ is the forest $\mathbb{T}_1 \cdot \mathbb{T}_2$ with decoration $\ell|_{\mathbb{T}_1} = \ell_1$, $\ell|_{\mathbb{T}_2} = n_1 + \ell_2$ (ℓ_i = decoration of \mathbb{T}_i , $i = 1, 2$), and $\mu_1 \otimes \mu_2 \in \tilde{\mathcal{P}}^{+, \mathbb{T}_1 \wedge \mathbb{T}_2} \text{Meas}(\mathbb{R}^{n_1+n_2})$ is the tensor measure

$$\mu_1 \otimes \mu_2(dx_1, \dots, dx_{n_1+n_2}) = \mu_1(dx_1, \dots, dx_{n_1}) \mu_2(dx_{n_1+1}, \dots, dx_{n_1+n_2}).$$

(iv) Let, for $\Gamma = (\Gamma(1), \dots, \Gamma(d))$, $\bar{\chi}_{\Gamma}^t : \mathbf{Sh}^d \rightarrow \mathbb{R}$ be the linear form on \mathbf{Sh}^d defined by

$$\bar{\chi}_{\Gamma}^t(\mathbb{T}_n) := \sum_{\sigma \in \Sigma_n} \phi_{\mu_{\Gamma}^{\sigma}}^t(\mathbb{T}^{\sigma}), \quad (3.18)$$

where $-\ell$ being the decoration of \mathbb{T}_n – one has set $\mu_{\Gamma} := \otimes_{j=1}^n d\Gamma(\ell(j))$, and \mathbb{T}^{σ} is the permutation graph associated to σ (see subsection 1.2).

Remarks.

1. Note that the **H**-multiplicative property (3.17) holds in particular for $\phi_{\mathbb{T}}^t = [\text{SkI}_{\mathbb{T}}(\cdot)]_t$ or $[\mathcal{R}\text{SkI}_{\mathbb{T}}(\cdot)]_t$, either trivially or by construction (see Step 4 in the construction of section 2). Note that $[\mathcal{R}\text{SkI}_{\mathbb{T}}(\mu)]_t$ has been defined *only* if $\mu \in \tilde{\mathcal{P}}^+ \text{Meas}(\mathbb{R}^n)$. If $\phi_{\mathbb{T}}^t = [\text{SkI}_{\mathbb{T}}(\cdot)]_t$, then simply $\bar{\chi}_{\Gamma}^t(\mathbb{T}_n) = [\text{SkI}_{\mathbb{T}_n}(\Gamma)]_t$ by the measure splitting lemma.
2. Assume $\mu_i \in \tilde{\mathcal{P}}^+ \text{Meas}(\mathbb{R}^{n_i}) \subset \tilde{\mathcal{P}}^{+, \mathbb{T}} \text{Meas}(\mathbb{R}^{n_i})$, where $\tilde{\mathcal{P}}^+$ is the $\tilde{\mathcal{P}}$ -projection associated to the subset $\mathbb{Z}_+^{n_i} := \{\mathbf{k} = (k_1, \dots, k_{n_i}); |k_1| \leq \dots \leq |k_{n_i}|\} \ (i = 1, 2)$. Then $\mu_1 \otimes \mu_2 \in \tilde{\mathcal{P}}^{+, \mathbb{T}_1 \wedge \mathbb{T}_2} \text{Meas}(\mathbb{R}^{n_1+n_2})$ but $\mu_1 \otimes \mu_2 \notin \tilde{\mathcal{P}}^+ \text{Meas}(\mathbb{R}^{n_1+n_2})$ in general; the product measure $\mu_1 \otimes \mu_2$ decomposes as a sum over shuffles ε of $(1, \dots, n_1), (n_1+1, \dots, n_1+n_2)$, namely, $\mu_1 \otimes \mu_2 = \sum_{\varepsilon \text{ shuffle}} (\mu_1 \otimes \mu_2)^{\varepsilon} \circ \varepsilon$. Hence the **H**-multiplicative property (3.17) reads also

$$\phi_{\mu_1}^t(\mathbb{T}_1) \phi_{\mu_2}^t(\mathbb{T}_2) = \sum_{\varepsilon \text{ shuffle}} \phi_{(\mu_1 \otimes \mu_2)^{\varepsilon}}^t(\varepsilon^{-1}(\mathbb{T}_1 \wedge \mathbb{T}_2)), \quad (3.19)$$

where $\varepsilon^{-1}(\mathbb{T}_1 \wedge \mathbb{T}_2)$ is the forest $\mathbb{T}_1 \wedge \mathbb{T}_2$ with decoration $\varepsilon^{-1} \circ \ell$, see Definition 3.4 (iii) for the definition of ℓ .

3. The regularization algorithm \mathcal{R} presented in section 2 may be written in a compact way using the structures we have just introduced. Namely, one has:

Lemma 3.5 *Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ and $\mu_{\Gamma} := \otimes_{j=1}^n d\Gamma(\ell(j))$. Then*

$$[\mathcal{R}\Gamma^n(\ell(1), \dots, \ell(n))]_{ts} = \sum_{\sigma \in \Sigma_n} (\phi^t * (\phi^s \circ S))_{\mu_{\Gamma}^{\sigma}}(\mathbb{T}^{\sigma}), \quad (3.20)$$

where

$$\phi_{\nu}^t(\mathbb{T}) := [\mathcal{R}\text{SkI}_{\mathbb{T}}(\nu)]_t = \left[\text{SkI}_{\mathbb{T}} \left(\sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}} (\otimes_{v \in V(\mathbb{T})} D(\phi_{k_v})) \nu \right) \right]_t \quad (3.21)$$

for $\nu \in \tilde{\mathcal{P}}^{+, \mathbb{T}} \text{Meas}(\mathbb{R}^n)$, and $(\phi^t * (\phi^s \circ S))_{\mu_{\sigma}}$ is the obvious multilinear extension of the convolution, see eq. (3.7).

Proof. Simple formalization of the regularization procedure explained in Section 2. \square

The fundamental result is the following.

Lemma 3.6 *Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be compactly supported, and assume that the condition of Step 1 in section 2 is satisfied. Then $\bar{\chi}_\Gamma^t$ is a character of \mathbf{Sh}^d .*

Proof. Let $\mathbb{T}_{n_i} \in \mathbf{Sh}^d$ with n_i vertices ($i = 1, 2$); define $n := n_1 + n_2$. Let $\mu_i := \otimes_{j=1}^{n_i} d\Gamma(\ell_i(j))$, $i = 1, 2$ and $\mu := \mu_1 \otimes \mu_2$. If $n' \geq 1$, we let $\mathbb{T}'_{n'}$ be the trunk tree with n' vertices $\{n' \rightarrow \dots \rightarrow 1\}$ and decoration $\ell(j) = j$, $j \leq n'$, see Fig. 1. All shuffles ε below are intended to be shuffles of $(1, \dots, n_1), (n_1 + 1, \dots, n_2)$. Then

$$\begin{aligned} \bar{\chi}_\Gamma^t(\mathbb{T}_{n_1} \circ \mathbb{T}_{n_2}) &= \sum_{\varepsilon \text{ shuffle}} \bar{\chi}_{\mu \circ \varepsilon}^t(\mathbb{T}'_n) \\ &= \sum_{\sigma \in \Sigma_n} \sum_{\varepsilon \text{ shuffle}} \phi_{(\mu \circ \varepsilon)^\sigma}^t(\mathbb{T}^\sigma) = \sum_{\sigma \in \Sigma_n} \sum_{\varepsilon \text{ shuffle}} \phi_{\mu^{\varepsilon \circ \sigma}}^t(\mathbb{T}^\sigma) \\ &=: \sum_{\sigma \in \Sigma_n} \phi_{\mu^\sigma}^t(\mathfrak{t}_1^\sigma) \end{aligned} \quad (3.22)$$

with

$$\mathfrak{t}_1^\sigma := \sum_{\varepsilon \text{ shuffle}} \mathbb{T}^{\varepsilon^{-1} \circ \sigma} \in \mathbf{H}_{n,n}^+. \quad (3.23)$$

On the other hand,

$$\begin{aligned} \bar{\chi}_\Gamma^t(\mathbb{T}_{n_1}) \bar{\chi}_\Gamma^t(\mathbb{T}_{n_2}) &= \bar{\chi}_{\mu_1}^t(\mathbb{T}'_{n_1}) \bar{\chi}_{\mu_2}^t(\mathbb{T}'_{n_2}) \\ &= \sum_{\sigma_1 \in \Sigma_{n_1}, \sigma_2 \in \Sigma_{n_2}} \phi_{\mu_1^{\sigma_1}}^t(\mathbb{T}^{\sigma_1}) \phi_{\mu_2^{\sigma_2}}^t(\mathbb{T}^{\sigma_2}) \\ &= \sum_{\sigma_1 \in \Sigma_{n_1}, \sigma_2 \in \Sigma_{n_2}} \sum_{\varepsilon \text{ shuffle}} \phi_{(\mu_1^{\sigma_1} \otimes \mu_2^{\sigma_2})^\varepsilon}^t(\varepsilon^{-1}(\mathbb{T}^{\sigma_1} \wedge \mathbb{T}^{\sigma_2})) \end{aligned} \quad (3.24)$$

by (3.19)

$$= \sum_{\sigma \in \Sigma_n} \phi_{\mu^\sigma}^t(\mathfrak{t}_2^\sigma) \quad (3.25)$$

where

$$\mathfrak{t}_2^\sigma := \sum_{(\sigma_1, \sigma_2, \varepsilon); (\sigma_1 \otimes \sigma_2) \circ \varepsilon = \sigma} \varepsilon^{-1}(\mathbb{T}^{\sigma_1} \wedge \mathbb{T}^{\sigma_2}). \quad (3.26)$$

Hence $\bar{\chi}_\Gamma^t$ is a character of \mathbf{Sh} if and only if $\mathfrak{t}_1^\sigma = \mathfrak{t}_2^\sigma$ for every $\sigma \in \Sigma_n$; let us prove this. Extend first (3.22) and (3.25) by multilinearity from tensor measures $\mu_1 \otimes \mu_2$ to a general measure $\mu \in Meas(\mathbb{R}^n)$. By the

usual shuffle identity, $\text{SkI}_\Gamma^t(\mathbb{T}_{n_1} \cap \mathbb{T}_{n_2}) = \text{SkI}_\Gamma^t(\mathbb{T}_{n_1}) \cdot \text{SkI}_\Gamma^t(\mathbb{T}_{n_2})$, so (3.22) and (3.25) coincide for $\bar{\chi}^t = [\text{SkI}(\cdot)]_t$. Choose $\sigma \in \Sigma_n$. For any $\mu \in \text{Meas}(\mathbb{R}^n)$, one has

$$[\text{SkI}_{\mu^\sigma}(\mathfrak{t}_1^\sigma - \mathfrak{t}_2^\sigma)]_t = 0. \quad (3.27)$$

This fact implies actually that $\mathfrak{t}_1^\sigma = \mathfrak{t}_2^\sigma$. Let us first give an informal proof of this statement. To begin with, note that the fact that $[\text{SkI}_\Gamma(\mathfrak{t})]_t = 0$ for every smooth path Γ does not imply in itself that $\mathfrak{t} = 0$ if $\mathfrak{t} \in \mathbf{H}$ is arbitrary. Namely, the character $\text{SkI}_\Gamma^t : \mathbf{H} \rightarrow \mathbb{R}$ quotients out via the canonical projection $\Pi : \mathbf{H} \rightarrow \mathbf{Sh}$, see Proposition 3.1, into a character $\mathbf{Sh} \rightarrow \mathbb{R}$, by the tree shuffle property; one may actually prove that $\text{SkI}_\Gamma^t(\mathfrak{t}) = 0$ for every smooth path Γ if and only if $\mathfrak{t} \in \text{Ker}(\Pi)$. In our case, the elements of $\mathcal{F}_{n,n}^+$ are linearly independent modulo $\text{Ker}(\Pi)$ because the ordering of the labels $\ell(j)$, $j = 1, \dots, n$ is compatible with the tree ordering – which *prevents any possibility of shuffling* –, hence $\mathfrak{t}_1^\sigma - \mathfrak{t}_2^\sigma = 0$.

Let us now give a more formal argument. Let $\mathfrak{t}_1^\sigma - \mathfrak{t}_2^\sigma =: \sum_j a_j \mathfrak{t}_j$, $a_j \in \mathbb{Z}$, $\mathfrak{t}_j \in \mathcal{F}_{n,n}^+$ two-by-two distinct, and define

$$F_{\mathfrak{t}}(\xi_1, \dots, \xi_n) := \frac{1}{\prod_{v \in V(\mathfrak{t})} (\xi_v + \sum_{w \rightarrow v} \xi_w)} \quad (3.28)$$

if $\mathfrak{t} \in \mathcal{F}_{n,n}^+$. Applying Lemma 4.5 to $[\text{SkI}_{\mu_m}(\mathfrak{t}_j)]_t$ where $(\mu_m \circ \sigma)_{m \geq 1} \in \mathcal{P}^+ \text{Meas}(\mathbb{R}^n)$ is a sequence of measures whose Fourier transform converges weakly to the Dirac distribution $\delta_{(\xi_1, \dots, \xi_n)}$, one gets

$$\sum_j a_j F_{\mathfrak{t}_j}(\xi_1, \dots, \xi_n) = 0, \quad |\xi_1| \leq \dots \leq |\xi_n|. \quad (3.29)$$

Since the left-hand side of (3.29) is a rational function, the equation extends to arbitrary $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. Note that

$$\prod_{v \in V(\mathfrak{t}_j)} (\xi_v + \sum_{w \rightarrow v} \xi_w) = (\xi_1 + \sum_{w \rightarrow 1} \xi_w) F_{\check{\mathfrak{t}}_j}(\xi_2, \dots, \xi_n), \quad (3.30)$$

where $\check{\mathfrak{t}}_j := \text{Lea}_{\{1\}}(\mathfrak{t}_j)$ is \mathfrak{t}_j severed of the vertex 1, which is one of its roots. Let J_Ω , $\Omega \subset \{2, \dots, n\}$ be the subset of indices j such that $\{v \in \{1, \dots, n\}; v \rightarrow 1 \text{ in } \mathfrak{t}_j\} = \Omega$, i.e. such that the tree component of 1 in \mathfrak{t}_j has vertex set Ω . Take the residue at $-\sum_{w \in \Omega} \xi_w$ of the left-hand side of (3.29), considered as a function of ξ_1 . This gives:

$$\sum_{j \in J_\Omega} a_j F_{\check{\mathfrak{t}}_j}(\xi_2, \dots, \xi_n) = 0, \quad \Omega \subset \{2, \dots, n\}. \quad (3.31)$$

Shifting by -1 the indices of vertices of $\check{\mathfrak{t}}_j$ and the labels $\ell(v), v \in V(\check{\mathfrak{t}}_j)$, one gets a forest in $\mathcal{F}_{n-1, n-1}^+$. One may now conclude by an inductive argument. \square

Let us now give an alternative definition for the regularization \mathcal{R} . As we shall see in Lemma 3.8, the two definitions actually coincide.

Definition 3.7 (alternative definition for regularization \mathcal{R}') *Choose for every tree $\mathbb{T} \in \mathbf{H}$ a subset $\mathbb{Z}_{reg}^{\mathbb{T}} \subset \mathbb{Z}_+^{\mathbb{T}}$ satisfying the condition stated in Step 1 of section 2. Let $\Gamma = (\Gamma(1), \dots, \Gamma(d))$ be a compactly supported, α -Hölder path, and $\mu_\Gamma := \otimes_{j=1}^n d\Gamma(\ell(j))$ the corresponding measure.*

(i) *Let, for every $\mathbb{T} \in \mathbf{H}^d$ with n vertices,*

$$\phi_\nu^t(\mathbb{T}) = [\mathcal{R}\text{SkI}_\mathbb{T}(\nu)]_t, \quad \nu \in \tilde{\mathcal{P}}^{+, \mathbb{T}} \text{Meas}(\mathbb{R}^n) \quad (3.32)$$

see eq. (2.11) or Lemma 3.5, and

$$\bar{\chi}_\Gamma^t(\mathbb{T}_n) := \sum_{\sigma \in \Sigma_n} \phi_{\mu_\Gamma^\sigma}^t(\mathbb{T}^\sigma) \quad (3.33)$$

be the associated character of \mathbf{Sh} as in Definition 3.4.

(ii) *Let, for $\mathbb{T}_n \in \mathbf{Sh}^d$, $n \geq 1$, with n vertices and decoration ℓ ,*

$$[\mathcal{R}'\mathbf{T}^n(\ell(1), \dots, \ell(n))]_{ts} := \bar{\chi}_\Gamma^t * (\bar{\chi}_\Gamma^s \circ \bar{S})(\mathbb{T}_n). \quad (3.34)$$

Since $\bar{\chi}_\Gamma^s$, $\bar{\chi}_\Gamma^t$ and hence $\bar{\chi}_\Gamma^t * (\bar{\chi}_\Gamma^s \circ \bar{S})$ are characters of the shuffle algebra, $\mathcal{R}'\mathbf{T}$ satisfies the *shuffle property*. Also, $\mathcal{R}'\mathbf{T}$ satisfies the *Chen property* by construction, since

$$\begin{aligned} [\mathcal{R}'\mathbf{T}^n(\ell(1), \dots, \ell(n))]_{ts} &= (\bar{\chi}_\Gamma^t * (\bar{\chi}_\Gamma^u \circ \bar{S})) * (\bar{\chi}_\Gamma^u * (\bar{\chi}_\Gamma^s \circ \bar{S}))(\mathbb{T}_n) \\ &= [\mathcal{R}'\mathbf{T}^n(\ell(1), \dots, \ell(n))]_{tu} + [\mathcal{R}'\mathbf{T}^n(\ell(1), \dots, \ell(n))]_{us} \\ &\quad + \sum_j [\mathcal{R}'\mathbf{T}^j(\ell(1), \dots, \ell(j))]_{tu} [\mathcal{R}'\mathbf{T}^{n-j}(\ell(j+1), \dots, \ell(n))]_{us} \end{aligned} \quad (3.35)$$

by definition of the convolution in \mathbf{Sh} . Both properties remain valid if $\bar{\chi}_\Gamma^t$, $t \in \mathbb{R}$ are arbitrary characters of \mathbf{Sh} .

Let us make this definition a little more explicit before proving that $\mathcal{R}' = \mathcal{R}$. Replacing $\bar{\chi}^s \circ \bar{S}$ with $\chi^s \circ S$, see Corollary 3.2, one gets, see eq.

(3.8),

$$\begin{aligned}
[\mathcal{R}'\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts} &= \chi_{\Gamma}^t(\mathbb{T}_n) + \chi_{\Gamma}^s(S(\mathbb{T}_n)) + \sum_j \chi_{\Gamma}^t(Roo_j \mathbb{T}_n)(\chi_{\Gamma}^s \circ S)(Lea_j \mathbb{T}_n) \\
&= (\bar{\chi}_{\Gamma}^t - \bar{\chi}_{\Gamma}^s)(\mathbb{T}_n) + \sum_j (\bar{\chi}_{\Gamma}^t - \bar{\chi}_{\Gamma}^s)(Roo_j \mathbb{T}_n) \cdot \chi_{\Gamma}^s(S(Lea_j \mathbb{T}_n)).
\end{aligned} \tag{3.36}$$

Expanding the formula for $S(Lea_j \mathbb{T}_n)$ in terms of multiple cuts as in the previous subsection, see eq. (3.9), we get

$$\begin{aligned}
[\mathcal{R}'\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts} &= (\bar{\chi}_{\Gamma}^t - \bar{\chi}_{\Gamma}^s)(\mathbb{T}_n) + \sum_{l \geq 1} (-1)^l \\
&\quad \sum_{j_1 < \dots < j_l} (\bar{\chi}_{\Gamma}^t - \bar{\chi}_{\Gamma}^s)(Roo_{j_1} \mathbb{T}_n) \left\{ \prod_{m=1}^{l-1} \bar{\chi}_{\Gamma}^s(Lea_{j_m} \circ Roo_{j_{m+1}}(\mathbb{T}_n)) \right\} \bar{\chi}_{\Gamma}^s(Lea_{j_l} \mathbb{T}_n),
\end{aligned} \tag{3.37}$$

by chopping the trunk tree \mathbb{T}_n . Finally, $\bar{\chi}_{\Gamma}^u(\mathbb{T})$, $u = t$ or s , should be split according to Definition 3.4 (iv).

Let us now make the following remark. The difference between $[\mathcal{R}\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts}$ and $[\mathcal{R}'\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts}$ is that $[\mathcal{R}\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts}$ is obtained by first (i) splitting the measure $\mu := \otimes_{j=1}^n d\Gamma(\ell(j))$ into $\sum_{\sigma \in \Sigma_n} \mu^{\sigma} \circ \sigma$ and then (ii) chopping the forests \mathbb{T}_j^{σ} , while $[\mathcal{R}'\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts}$ is obtained by first (i) chopping the trunk tree \mathbb{T}_n and then (ii) splitting the measures on the trunk subtrees. Actually, as may be expected, the two operations commute.

Lemma 3.8 $[\mathcal{R}'\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts} = [\mathcal{R}\mathbf{\Gamma}^n(\ell(1), \dots, \ell(n))]_{ts}$. Hence the regularized iterated integrals $\mathcal{R}\mathbf{\Gamma}$ satisfy the Chen and shuffle properties, and Theorem 3.1 is proved.

Proof. The proof goes along the same lines as Lemma 3.6. Let \mathbb{T}_n be some trunk tree with n vertices and decoration ℓ , and $\mu := \otimes_{j=1}^n d\Gamma(\ell(j))$. Consider for the moment an arbitrary character $\bar{\chi}_{\Gamma}^t$ as in Lemma 3.6, associated to linear forms $\phi_{\mathbb{T}}^t$ as in Definition 3.4. Define quite generally

$$[\mathcal{R}_{\phi, \Gamma}(\mathbb{T}_n)]_{ts} := \sum_{\sigma \in \Sigma_n} (\phi^t * (\phi^s \circ S))_{\mu^{\sigma}}(\mathbb{T}^{\sigma}), \tag{3.38}$$

see Lemma 3.5, and (see Definition 3.7 (ii))

$$[\mathcal{R}'_{\phi,\Gamma}(\mathbb{T}_n)]_{ts} := (\chi_\Gamma^t * (\chi_\Gamma^s \circ S))(\mathbb{T}_n). \quad (3.39)$$

If $\phi_\mathbb{T}^t = [\mathcal{R}\text{SkI}_\mathbb{T}(\cdot)]_t$, then $\mathcal{R}_{\phi,\Gamma} = \mathcal{R}\Gamma^n$ and $\mathcal{R}'_{\phi,\Gamma} = \mathcal{R}'\Gamma^n$. On the other hand, if $\phi_\Gamma^t = [\text{SkI}_\mathbb{T}(\cdot)]_t$, then plainly $[\mathcal{R}_{\phi,\Gamma}(\mathbb{T}_n)]_{ts} = [\mathcal{R}'_{\phi,\Gamma}(\mathbb{T}_n)]_{ts} = [\Gamma^{cano,n}(\ell(1), \dots, \ell(n))]_{ts}$, see first comment in section 2 and eq. (3.12).

Let $\sigma \in \Sigma_n$. Fix some multi-index $\mathbf{k} = (k_1, \dots, k_n)$ such that $|k_1| \leq \dots \leq |k_n|$, and set $\mu_{\mathbf{k}}^\sigma = \tilde{\mathcal{P}}^{\{\mathbf{k}\}}(\mu \circ \sigma)$. Then, see eq. (3.7)

$$\begin{aligned} (\phi^t * (\phi^s \circ S))_{\mu_{\mathbf{k}}^\sigma}(\mathbb{T}^\sigma) &= \frac{1}{|\Sigma_{\mathbf{k}}|} \left(\phi_{\mu_{\mathbf{k}}^\sigma}^t(\mathbb{T}^\sigma) + \phi_{\mu_{\mathbf{k}}^\sigma}^s(S(\mathbb{T}^\sigma)) + \right. \\ &\quad \left. + \sum_{\mathbf{v} \models V(\mathbb{T}^\sigma)} \phi_{\text{Roo}_{\mathbf{v}}(\mu_{\mathbf{k}}^\sigma)}^t(\text{Roo}_{\mathbf{v}}(\mathbb{T}^\sigma)) \phi_{\text{Lea}_{\mathbf{v}}(\mu_{\mathbf{k}}^\sigma)}^s(S(\text{Lea}_{\mathbf{v}}(\mathbb{T}^\sigma))) \right) \end{aligned} \quad (3.40)$$

Expand S according to eq. (3.9). This gives an expression for the $\tilde{\mathcal{P}}^{\{\mathbf{k}\}}$ -projection of $[\mathcal{R}_{\phi,\Gamma}(\mathbb{T}_n)]_{ts}$. An expression may also be obtained for the analogous regularized quantity associated to \mathcal{R}' by using eq. (3.37). In the end, one gets two sums over some subsets of $\{1, \dots, n\}$,

$$[\mathcal{R}_{\phi,\mathcal{P}^{\{\mathbf{k} \circ \sigma\}}\Gamma}(\mathbb{T}_n)]_{ts} = \sum_{J \subset \{1, \dots, n\}} \sum_j \phi_{\mu_{\mathbf{k}}^\sigma|_J}^t(\mathbf{t}_{1,J,j}^\sigma) \phi_{\mu_{\mathbf{k}}^\sigma|_{\bar{J}}}^s(\mathbf{t}'_{1,\bar{J},j}{}^\sigma) \quad (3.41)$$

and similarly

$$[\mathcal{R}'_{\phi,\mathcal{P}^{\{\mathbf{k} \circ \sigma\}}\Gamma}(\mathbb{T}_n)]_{ts} = \sum_{J \subset \{1, \dots, n\}} \sum_j \phi_{\mu_{\mathbf{k}}^\sigma|_J}^t(\mathbf{t}_{2,J,j}^\sigma) \phi_{\mu_{\mathbf{k}}^\sigma|_{\bar{J}}}^s(\mathbf{t}'_{2,\bar{J},j}{}^\sigma) \quad (3.42)$$

where:

$$J = V(\mathbf{t}_{1,J,j}^\sigma) = V(\mathbf{t}_{2,J,j}^\sigma), \quad \bar{J} = \{1, \dots, n\} \setminus J = V(\mathbf{t}'_{1,\bar{J},j}{}^\sigma) = V(\mathbf{t}'_{2,\bar{J},j}{}^\sigma); \quad (3.43)$$

$$\mu_{\mathbf{k}}^\sigma|_J = \otimes_{1 \leq j \leq n, j \in J} d\tilde{\mathcal{P}}^{\{\mathbf{k} \circ \sigma\}}\Gamma(\ell(j)), \quad \mu_{\mathbf{k}}^\sigma|_{\bar{J}} = \otimes_{1 \leq j \leq n, j \in \bar{J}} d\tilde{\mathcal{P}}^{\{\mathbf{k} \circ \sigma\}}\Gamma(\ell(j)); \quad (3.44)$$

and $\mathbf{t}_{1,J,j}^\sigma, \mathbf{t}'_{1,\bar{J},j}{}^\sigma, \mathbf{t}_{2,J,j}^\sigma, \mathbf{t}'_{2,\bar{J},j}{}^\sigma \in \mathcal{F}_{n,n}^+$ as in the proof of Lemma 3.6.

In the case of the regularization scheme \mathcal{R} , each $\mathbf{t}_{1,J,j}^\sigma$ is a forest such as $\text{Roo}_{\mathbf{v}}(\mathbb{T}_l^\sigma)$, where \mathbb{T}_l^σ appears in the decomposition of the permutation graph \mathbb{T}^σ , and \mathbf{v} is some admissible cut of \mathbb{T}_l^σ , while $\mathbf{t}'_{1,\bar{J},j}{}^\sigma$ is some complicated product obtained by the multiple cut decomposition of $S(\text{Lea}_{\mathbf{v}}(\mathbb{T}_l^\sigma))$. In the

case of \mathcal{R}' , one first splits \mathbb{T}_n into $(\text{Root}\mathbb{T}_n, \text{Leaf}\mathbb{T}_n)$ and then permutes the vertices of each of the two trunk subtrees, see eq. (3.37).

As in the proof of Lemma 3.6, one now proves the equality

$$\sum_J \sum_j \mathfrak{t}_{1,J,j}^\sigma \otimes \mathfrak{t}_{1,\bar{J},j}^{\prime\sigma} = \sum_J \sum_j \mathfrak{t}_{2,J,j}^\sigma \otimes \mathfrak{t}_{2,\bar{J},j}^{\prime\sigma} \quad (3.45)$$

by assuming that $\phi_{\mathbb{T}}^t = [\text{SkI}_{\mathbb{T}}(\cdot)]_t$, in which case both expressions (3.41) and (3.42) are equal. By considering a sequence of measures $(\mu_m \circ \sigma)_{m \geq 1}$ whose Fourier transforms converge weakly to $\delta_{(\xi_1, \dots, \xi_n)}$, one gets by Lemma 4.5 an equation of the type

$$\begin{aligned} \sum_J \left[e^{i(s \sum_{i \in J} \xi_i + t \sum_{i \in \bar{J}} \xi_i)} \sum_j F_{\mathfrak{t}_{1,J,j}^\sigma}((\xi_i)_{i \in J}) F_{\mathfrak{t}_{1,\bar{J},j}^{\prime\sigma}}((\xi_i)_{i \in \bar{J}}) \right. \\ \left. + \sum_J e^{i(s \sum_{i \in J} \xi_i + t \sum_{i \in \bar{J}} \xi_i)} \sum_j F_{\mathfrak{t}_{2,J,j}^\sigma}((\xi_i)_{i \in J}) F_{\mathfrak{t}_{2,\bar{J},j}^{\prime\sigma}}((\xi_i)_{i \in \bar{J}}) \right] = 0. \end{aligned} \quad (3.46)$$

where the function $F_{\mathfrak{t}}$ has been defined in the course of the proof of Lemma 3.6. Under the generic condition that all $\xi_J := \sum_{i \in J} \xi_i$, $J \subset \{1, \dots, n\}$ are two-by-two distinct, the functions $(s, t) \mapsto f_J(s, t) := e^{i(s\xi_J + t\xi_{\bar{J}})}$, $J \subset \{1, \dots, n\}$ are linearly independent. Hence, for every J ,

$$\sum_j F_{\mathfrak{t}_{1,J,j}^\sigma}((\xi_i)_{i \in J}) F_{\mathfrak{t}_{1,\bar{J},j}^{\prime\sigma}}((\xi_i)_{i \in \bar{J}}) + F_{\mathfrak{t}_{2,J,j}^\sigma}((\xi_i)_{i \in J}) F_{\mathfrak{t}_{2,\bar{J},j}^{\prime\sigma}}((\xi_i)_{i \in \bar{J}}) = 0. \quad (3.47)$$

By using the same arguments as in the proof of Lemma 3.6, one obtains eq. (3.45). \square

4 Hölder estimates

Let Γ be an α -Hölder path. We shall now *choose a regularization scheme*, i.e. choose for each tree \mathbb{T} a subset $\mathbb{Z}_{reg}^\mathbb{T} \subset \mathbb{Z}_+^\mathbb{T}$ such that the convergence condition stated in section 2, Step 1 is verified, and prove that the associated regularized rough path $\mathcal{R}\Gamma^n(\ell(1), \dots, \ell(n))$ satisfies the required Hölder properties. Following the regularization procedure as explained in section 2, one must first (1) decompose $\mathcal{R}\Gamma^n(\ell(1), \dots, \ell(n))$ into the sum over all permutations $\sigma \in \Sigma_n$ of $\mathcal{R}I_{\mathbb{T}_j^\sigma}^{\tilde{\mathcal{P}}^+} \left(\otimes_{v \in V(\mathbb{T}_j^\sigma)} \Gamma(\ell(\sigma(v))) \right)$ as in the final step

of section 2, and (2) show Hölder regularity with correct exponent of the *increment terms* $\mathcal{R}\text{SkI}_{\mathbb{T}}(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v))))$ and of the *boundary terms*, $\mathcal{R}I_{\mathbb{T}}(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v))))(\partial)$, see Step 4.

4.1 Choice of the regularization scheme

Recall that the whole algorithm rests on the *choice* of a subdomain $\mathbb{Z}_{reg}^{\mathbb{T}} \subset \mathbb{Z}_+^{\mathbb{T}} := \{(k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}} \mid (v \rightarrow w) \Rightarrow |k_v| \geq |k_w|\}$ for each tree $\mathbb{T} \in \mathcal{T}$. The purpose of this subsection is to propose an adequate choice.

We shall first need to introduce a little more terminology concerning tree structures (see Fig. 5).

Definition 4.1 *Let \mathbb{T} be a tree.*

- (i) *A vertex v is a leaf if no vertex connects to v . The set of leaves above (i.e. connecting to) $v \in V(\mathbb{T})$ is denoted by $Leaf(v)$.*
- (ii) *Vertices at which 2 or more branches join are called nodes.*
- (iii) *The set $Br(v_1 \rightarrow v_2)$ of vertices from a leaf or a node v_1 to a node v_2 or to the root, is called a branch if it does not contain any other node. By convention, $Br(v_1 \rightarrow v_2)$ includes v_1 and excludes v_2 .*
- (iv) *A node n is called an uppermost node if no other node is connected to n .*

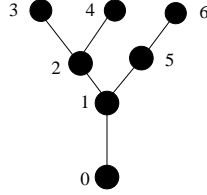


Figure 5: 3,4,6 are leaves; 1, 2 and 5 are nodes, 2 and 5 are uppermost; branches are e.g. $Br(2 \rightarrow 1) = \{2\}$ or $Br(6 \rightarrow 1) = \{6, 5\}$; $Leaf(2) = \{3, 4\}$; $w_{max}(2) = 4$.

Definition 4.2 *Let \mathbb{T} be a tree. If $v \in V(\mathbb{T})$, we let $w_{max}(v) := \max\{w \in V(\mathbb{T}) \mid w \rightarrow v\}$, or simply $w_{max}(v) = v$ if v is a leaf.*

Definition 4.3 *Let $\mathbb{Z}_{reg}^{\mathbb{T}}$ be the set of $V(\mathbb{T})$ -uples $\mathbf{k} = (k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}}$ such that the following conditions are satisfied:*

- (i) if $v < w$, then $|k_v| \leq |k_w|$;
- (ii) if $v \in V(\mathbb{T})$ and $w \in \text{Leaf}(v)$, $k_w \cdot k_v < 0$, then $|k_v| \leq |k_w| - \log_2 10 - \log_2 |V(\mathbb{T})|$;
- (iii) if $n \in V(\mathbb{T})$ is a node, then each vertex $w \in \{w_{\max}(v) \mid v \rightarrow n\}$ such that $k_w \cdot k_{w_{\max}(n)} < 0$ satisfies: $|k_w| \leq |k_{w_{\max}(n)}| - \log_2 10 - \log_2 |V(\mathbb{T})|$.

Lemma 4.4 Let $\xi = (\xi_v)_{v \in \mathbb{T}}$ such that $\xi_v \in \text{supp}(\phi_{k_v})$ for some $\mathbf{k} = (k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}_{\text{reg}}^{\mathbb{T}}$, where $(\phi_k)_{k \in \mathbb{Z}}$ is the dyadic partition of unity defined in the Appendix. Then, for every $v \in V$,

$$|V(\mathbb{T})| \cdot |\xi_{w_{\max}(v)}| \geq |\xi_v + \sum_{w \rightarrow v} \xi_w| > \frac{1}{2} |\xi_{w_{\max}(v)}|. \quad (4.1)$$

Proof.

The left inequality is trivial. As for the right one, assume first that v is on a terminal branch, i.e. $\text{Leaf}(v) = \{w_{\max}(v)\}$ is a singleton. Then Definition 4.3 (ii) implies the following: for every vertex v' on the branch between $w_{\max}(v)$ and v , i.e. $v' \in \text{Br}(w_{\max}(v) \rightarrow v) \cup \{v\}$,

- either $\xi_{v'}$ is of the same sign as $\xi_{w_{\max}(v)}$;
- or $|\xi_{v'}| \leq \frac{|\xi_{w_{\max}(v)}|}{2|V(\mathbb{T})|}$ since $|\xi_{v'}| \in (2^{|k_{v'}|-1}, 5 \cdot 2^{|k_{v'}|-1})$ (and similarly for $|\xi_{w_{\max}(v)}|$) by the remarks following Proposition 5.2.

Hence $|\xi_v + \sum_{w \rightarrow v} \xi_w| = |\sum_{v' \in \text{Br}(w_{\max}(v) \rightarrow v) \cup \{v\}} \xi_{v'}| > \left(1 - \frac{1}{2} \frac{|\{w : w \rightarrow v\}|}{|V(\mathbb{T})|}\right) |\xi_{w_{\max}(v)}|$ and $\xi_v + \sum_{w \rightarrow v} \xi_w$ has same sign as $\xi_{w_{\max}(v)}$.

Consider now what happens at a node n . Let $n^+ := \{v \in V(\mathbb{T}) \mid v \rightarrow n\}$. Assume by induction on the number of vertices that, for all $v \in n^+$,

$$(1 + |\{w : w \rightarrow v\}|) |\xi_{w_{\max}(v)}| \geq |\xi_v + \sum_{w \rightarrow v} \xi_w| > \left(1 - \frac{1}{2} \frac{|\{w : w \rightarrow v\}|}{|V(\mathbb{T})|}\right) \cdot |\xi_{w_{\max}(v)}| \quad (4.2)$$

and that $\xi_v + \sum_{w \rightarrow v} \xi_w$ has same sign as $\xi_{w_{\max}(v)}$. By Definition 4.3 (iii), either $\xi_{w_{\max}(v)} \cdot \xi_{w_{\max}(n)} > 0$ or $|\xi_{w_{\max}(v)}| \leq \frac{|\xi_{w_{\max}(n)}|}{2|V(\mathbb{T})|}$. Then, letting w_0 be

the element of n^+ such that $w_{\max}(v_0) = w_{\max}(n)$,

$$\begin{aligned}
(1 + |\{w : w \rightarrow n\}|) |\xi_{w_{\max}(n)}| &\geq |\xi_n + \sum_{w \rightarrow n} \xi_w| = \left| \xi_n + \sum_{v \in n^+} (\xi_v + \sum_{w \rightarrow v} \xi_w) \right| \\
&\geq \left| \xi_{v_0} + \sum_{w \rightarrow v_0} \xi_w \right| - \left| \sum_{v \in n^+; \xi_{w_{\max}(v)} \cdot \xi_{w_{\max}(n)} < 0} (\xi_v + \sum_{w \rightarrow v} \xi_w) \right| - |\xi_n| \\
&> \left(1 - \frac{1}{2} \frac{|\{w : w \rightarrow n\}|}{|V(\mathbb{T})|} \right) \cdot |\xi_{w_{\max}(n)}|. \tag{4.3}
\end{aligned}$$

□

4.2 A key formula for skeleton integrals

We assume in this paragraph that Γ is smooth and denote by Γ' its derivative. The Hölder estimates in subsections 4.3 and 4.4 rely on the key formula below.

Lemma 4.5 *The following formula holds:*

$$[\text{SkI}_{\mathbb{T}}(\Gamma)]_s = (i\sqrt{2\pi})^{-|V(\mathbb{T})|} \int \dots \int \prod_{v \in V(\mathbb{T})} d\xi_v \cdot e^{is \sum_{v \in V(\mathbb{T})} \xi_v} \frac{\prod_{v \in V(\mathbb{T})} \mathcal{F}(\Gamma'(\ell(v)))(\xi_v)}{\prod_{v \in V(\mathbb{T})} (\xi_v + \sum_{w \rightarrow v} \xi_w)}. \tag{4.4}$$

Proof. We use induction on $|V(\mathbb{T})|$. After stripping the root of \mathbb{T} , denoted by 0, there remains a forest $\mathbb{T}' = \mathbb{T}'_1 \dots \mathbb{T}'_J$, whose roots $0_1, \dots, 0_J$ are the vertices directly connected to 0. Assume

$$[\text{SkI}_{\mathbb{T}'_j}(\Gamma)]_{x_0} = \int \dots \int \prod_{v \in V(\mathbb{T}'_j)} d\xi_v \cdot e^{ix_0 \sum_{v \in V(\mathbb{T}'_j)} \xi_v} F_j(\xi_{0_j}, (\xi_v)_{v \in \mathbb{T}'_j \setminus \{0_j\}}) \tag{4.5}$$

for some functions F_j , $j = 1, \dots, J$. Note that

$$\mathcal{F}(\text{SkI}_{\mathbb{T}'_j}(\Gamma))(\xi_j) = \left[\prod_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}} \int d\xi_v \right] F_j(\xi_j - \sum_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}} \xi_v, (\xi_v)_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}}). \tag{4.6}$$

Then

$$\begin{aligned}
[\text{SkI}_{\mathbb{T}}(\Gamma)]_s &= \int^s d\Gamma_{x_0}(\ell(0)) \prod_{j=1}^J [\text{SkI}_{\mathbb{T}'_j}(\Gamma)]_{x_0} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{d\xi}{i\xi} e^{is\xi} \mathcal{F} \left(\Gamma'(\ell(0)) \prod_{j=1}^J \text{SkI}_{\mathbb{T}'_j}(\Gamma) \right) (\xi) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\xi \mathcal{F}(\Gamma'(\ell(0))) (\xi - \sum_{j=1}^J \xi_j) \frac{e^{is\xi}}{i\xi} \cdot \int d\xi_1 \dots \int d\xi_J \\
&\quad \left[\prod_{j=1}^J \prod_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}} \int d\xi_v \right] \prod_{j=1}^J F_j(\xi_j - \sum_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}} \xi_v, (\xi_v)_{v \in V(\mathbb{T}'_j) \setminus \{0_j\}})
\end{aligned} \tag{4.7}$$

hence the result. \square

4.3 Estimate for the increment term

We now come back to an arbitrary α -Hölder path and prove a Hölder estimate for the increment term, see eq. (2.13), which is simply a regularized skeleton integral.

Let $\sigma \in \Sigma_n$ be a permutation, and \mathbb{T} be one of the forests \mathbb{T}_j^σ appearing in the permutation graph \mathbb{T}^σ , see Lemma 1.5. Hölder norms $\|\cdot\|_{\mathcal{C}^\gamma}$ are defined in the Appendix. Recall \mathbb{T} comes with a total ordering compatible with its tree partial ordering. The $\tilde{\mathcal{P}}$ -projection $\tilde{\mathcal{P}}^+$ below is defined with respect to this total ordering.

Lemma 4.6 (Hölder estimate of the increment term) *It holds*

$$\|\mathcal{R}\text{SkI}_{\mathbb{T}} \left(\tilde{\mathcal{P}}^+ (\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) \|_{\mathcal{C}^{\lfloor V(\mathbb{T}) \rfloor \alpha}} < \infty. \tag{4.8}$$

Remark. Although formal integral integrals are a priori infra-red divergent (see subsection 1.4), the formula given in Lemma 4.5 for skeleton integrals delivers infra-red convergent quantities when one restricts the integration over $\xi = (\xi_v)_{v \in V(\mathbb{T})}$ to the subdomain associated to $\mathbb{Z}_{reg}^{\mathbb{T}}$, see Lemma 4.4, because

$$\left| \frac{\mathcal{F}(\Gamma'(\ell(v)))(\xi_v)}{\xi_v + \sum_{w \rightarrow v} \xi_w} \right| \lesssim |\mathcal{F}(\Gamma(\ell(v)))(\xi_v)| \frac{|\xi_v|}{|\xi_{w_{max}(v)}|} \leq |\mathcal{F}(\Gamma(\ell(v)))(\xi_v)| \tag{4.9}$$

is bounded.

Proof.

We implicitly assume in the proof that \mathbb{T} is a tree, leaving the obvious generalization to forests with several components to the reader.

We shall start the computations by adapting the proof of a theorem in [30], §2.6.1 bounding the Hölder-Besov norm of the product of two Hölder functions. Write

$$G(x) = \left[\mathcal{R}\text{SkI}_{\mathbb{T}} \left(\tilde{\mathcal{P}}^+ (\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) \right]_x. \quad (4.10)$$

By Lemma 4.5,

$$\begin{aligned} G(x) &= (i\sqrt{2\pi})^{-|V(\mathbb{T})|} \sum_{\mathbf{k}=(k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}_{reg}^{\mathbb{T}}} \int_{\prod_{v \in V(\mathbb{T})} \text{supp}(\phi_{k_v})} \prod_{v \in V(\mathbb{T})} d\xi_v \cdot \\ &\cdot e^{ix \sum_{v \in V(\mathbb{T})} \xi_v} \frac{\prod_{v \in V(\mathbb{T})} \mathcal{F}(D(\phi_{k_v})\Gamma'(\ell(\sigma(v))))(\xi_v)}{\prod_{v \in V(\mathbb{T})} (\xi_v + \sum_{w \rightarrow v} \xi_w)}. \end{aligned} \quad (4.11)$$

Write, for $\boldsymbol{\xi} = (\xi_v)_{v \in V(\mathbb{T})}$,

$$\Theta(\boldsymbol{\xi}) = \prod_{v \in V(\mathbb{T})} \frac{\xi_v}{\xi_v + \sum_{w \rightarrow v} \xi_w} \quad (4.12)$$

and

$$\Theta_1(\mathbf{k}) = \prod_{v \in V(\mathbb{T})} \frac{2^{|k_v|}}{2^{|k_{w_{max}(v)}|}}. \quad (4.13)$$

Let finally

$$\Theta_{\mathbf{k}}(\boldsymbol{\xi}) := \prod_{v \in V(\mathbb{T})} \sqrt{\phi_{k_v}}(\xi_v) \cdot \frac{\Theta(\boldsymbol{\xi})}{\Theta_1(\mathbf{k})}. \quad (4.14)$$

By Lemma 4.4, $\|\Theta_{\mathbf{k}}\|_{S^0(\mathbb{R}^{V(\mathbb{T})})}$, see Proposition 5.8, is *uniformly bounded* in \mathbf{k} if $\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}$, which is the key point for the following estimates.

Let $k \in \mathbb{Z}$. Apply the operator $D(\phi_k)$ to eq. (4.11): then, letting $\phi_k^*(\boldsymbol{\xi}) := \phi_k(\sum_{v \in V(\mathbb{T})} \xi_v)$,

$$D(\phi_k)G(x) = \left[\sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}} \Theta_1(\mathbf{k}) D(\Theta_{\mathbf{k}}) D(\phi_k^*) \cdot \prod_{v \in V(\mathbb{T})} D(\sqrt{\phi_{k_v}}) \Gamma(\ell(\sigma(v))) \right] (\mathbf{x}), \quad (4.15)$$

where $\mathbf{x} = (x_v)_{v \in V(\mathbb{T})} = (x, \dots, x)$ is a vector with $|V(\mathbb{T})|$ identical components.

Let $v_{max} := \sup\{v \mid v \in V(\mathbb{T})\}$. Note that $D(\phi_k^*) \cdot D(\otimes_{v \in V(\mathbb{T})} \sqrt{\phi_{k_v}})$ vanishes except if

$$\left(\sum_{v \in V(\mathbb{T})} \text{supp}(\phi_{k_v}) \right) \cap \text{supp}(\phi_k^*) \neq \emptyset, \quad (4.16)$$

which implies by Lemma 4.4

$$|k_{v_{max}} - k| = O(\log_2 |V(\mathbb{T})|); \quad (4.17)$$

namely, denoting by 0 the root of \mathbb{T} , $|V(\mathbb{T})| \cdot |\xi_{k_{v_{max}}}| \geq |\sum_{v \in V(\mathbb{T})} \xi_{k_v}| = |\xi_{k_0} + \sum_{w \rightarrow 0} \xi_{k_w}| > \frac{1}{2} |\xi_{k_{v_{max}}}|$ if $\xi_v \in \text{supp}(\phi_{k_v})$ for every v .

Since $\Theta_{\mathbf{k}}, \phi_k^* \in S^0(\mathbb{R}^{V(\mathbb{T})})$, one gets by Proposition 5.8

$$\|D(\phi_k)G\|_\infty \lesssim \sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}, k_{v_{max}}=k} \Theta_1(\mathbf{k}) \prod_{v \in V(\mathbb{T})} \|D(\sqrt{\phi_{k_v}})\Gamma(\ell(\sigma(v)))\|_\infty. \quad (4.18)$$

Since Γ is in \mathcal{C}^α , one obtains by Propositions 5.7 and 5.8:

$$\begin{aligned} \|D(\phi_k)G\|_\infty &\lesssim \sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}, k_{v_{max}}=k} \Theta_1(\mathbf{k}) \prod_{v \in V(\mathbb{T})} 2^{-|k_v|\alpha} \\ &\lesssim \sum_{\mathbf{k} \in \mathbb{Z}_{reg}^{\mathbb{T}}, k_{v_{max}}=k} \prod_{v \in V(\mathbb{T})} 2^{|k_v|(1-\alpha) - |k_{w_{max}(v)}|}. \end{aligned} \quad (4.19)$$

In other words, loosely speaking, each vertex $v \in V(\mathbb{T})$ contributes a factor $2^{|k_v|(1-\alpha) - |k_{w_{max}(v)}|}$ to $\|D(\phi_k)G\|_\infty$. If v is a leaf, then this factor is simply $2^{-|k_v|\alpha}$. Note that the upper bound $2^{|k_v|(1-\alpha) - |k_{w_{max}(v)}|} \leq 2^{-|k_v|\alpha}$ holds true for any vertex v .

Consider an uppermost node n , i.e. a node to which no other node is connected, together with the set of leaves $\{w_1 < \dots < w_J\}$ above n , see Fig. 5. Let $p_j = |V(Br(w_j \rightarrow n))|$. On the branch number j ,

$$2^{-|k_{w_j}|\alpha} \prod_{v \in Br(w_j \rightarrow n) \setminus \{w_j\}} \sum_{|k_v| \leq |k_{w_j}|} 2^{|k_v|(1-\alpha) - |k_{w_j}|} \lesssim 2^{-|k_{w_j}|\alpha p_j} \quad (4.20)$$

and (summing over $k_{w_1}, \dots, k_{w_{J-1}}$ and over k_n)

$$\begin{aligned}
& 2^{-|k_{w_J}| \alpha p_J} \sum_{|k_{w_{J-1}}| \leq |k_{w_J}|} 2^{-|k_{w_{J-1}}| \alpha p_{J-1}} \\
& \left(\dots \left(\sum_{|k_{w_1}| \leq |k_{w_2}|} 2^{-|k_{w_1}| \alpha p_1} \left(\sum_{|k_n| \leq |k_{w_1}|} 2^{|k_n|(1-\alpha) - |k_{w_J}|} \right) \right) \dots \right) \\
& \lesssim 2^{-|k_{w_J}| \alpha W(n)}, \tag{4.21}
\end{aligned}$$

where $W(n) = p_1 + \dots + p_J + 1 = |\{v : v \twoheadrightarrow n\}| + 1$ is the *weight* of n .

One may then consider the reduced tree \mathbb{T}_n obtained by shrinking all vertices above n (including n) to *one* vertex with weight $W(n)$ and perform the same operations on \mathbb{T}_n . Repeat this inductively until \mathbb{T} is shrunk to one point. In the end, one gets $\|D(\phi_k)G\|_\infty \lesssim 2^{-|k_{vmax}| \alpha |V(\mathbb{T})|} \lesssim 2^{-|k| \alpha |V(\mathbb{T})|}$, hence $G \in \mathcal{C}^{|V(\mathbb{T})| \alpha}$.

□

Remark. Note that the above proof breaks down for the non-regularized quantities, since the function $\Theta_k(\xi)$ is unbounded on $\mathbb{Z}_+^{\mathbb{T}} \setminus \mathbb{Z}_{reg}^{\mathbb{T}}$. For instance, the Lévy area of fractional Brownian motion diverges below the barrier $\alpha = 1/4$, see [11], [32], [33]. For deterministic, well-behaved paths Γ with very regular, polynomially decreasing Fourier components, the unregularized integrals are probably well-defined at least for $\alpha > 1/2$ – in which case the much simpler Young integral converges –, otherwise the case is not even clear.

4.4 Estimate for the boundary term

We shall now prove a Hölder estimate corresponding to the boundary term. As in the previous paragraph, we let $\sigma \in \Sigma_n$ and \mathbb{T} be one of the forests \mathbb{T}_j^σ , $j = 1, \dots, J_\sigma$. Once again, recall \mathbb{T} comes with a total ordering compatible with its tree partial ordering. The $\tilde{\mathcal{P}}$ -projection $\tilde{\mathcal{P}}^+$ below is defined with respect to this total ordering.

Lemma 4.7 (Hölder regularity of the boundary term) *The regularized boundary term $\left[\mathcal{R}I_{\mathbb{T}} \left(\tilde{\mathcal{P}}^+ (\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) (\partial) \right]_{ts}$ is $|V(\mathbb{T})| \alpha$ -Hölder.*

Proof.

As in the previous proof, we assume implicitly that \mathbb{T} is a tree, but the proof generalizes with only very minor changes to the case of forests.

Solving in terms of multiple cuts as in section 3 the recursive definition of the boundary term $[\mathcal{R}I_{\mathbb{T}} \left(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) (\partial)]_{ts}$ given in section 2, one gets in the end a sum of 'skeleton-type' terms of the form (see Fig. 6)

$$A_{ts} := [\delta \mathcal{R} \text{SkI}_{\text{Roo}(\mathbb{T})}]_{ts} \left(\prod_{m=1}^{l-1} [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_m} \circ \text{Roo}_{\mathbf{v}_{m+1}}(\mathbb{T})}]_s \right) [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_l}(\mathbb{T})}]_s \left(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) \quad (4.22)$$

where $\mathbf{v}_l = (v_{l,1} < \dots < v_{l,J_l}) \models V(\mathbb{T})$, $\mathbf{v}_{l-1} \models V(\text{Roo}_{\mathbf{v}_l} \mathbb{T})$, \dots , $\mathbf{v}_1 = (v_{1,1}, \dots, v_{1,J_1}) \models \text{Roo}_{\mathbf{v}_2}(\mathbb{T})$ and one has set for short $\text{Roo}(\mathbb{T}) := \text{Roo}_{\mathbf{v}_1}(\mathbb{T})$.

First step.

Let $U[\mathbf{k}] \subset \prod_{j=1}^{J_l} \mathbb{Z}_{reg}^{\text{Lea}_{v_{l,j}} \mathbb{T}}$ such that $\mathbf{k} = (k_{v_{l,1}}, \dots, k_{v_{l,J_l}})$ (with $|k_{v_{l,1}}| \leq \dots \leq |k_{v_{l,J_l}}|$) is fixed. Then (see after eq. (4.19) in the proof of Lemma 4.6) each vertex v contributes a factor $2^{|k_v|(1-\alpha)-|k_{w_{max}}(v)|} \leq 2^{-|k_v|\alpha}$, hence

$$\begin{aligned} \|\mathcal{P}^{U[\mathbf{k}]} \mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_l} \mathbb{T}}(\otimes_{v \in V(\text{Lea}_{\mathbf{v}_l} \mathbb{T})} \Gamma(\ell(\sigma(v))))\|_{\infty} &\lesssim \prod_{v \in \mathbf{v}_l} \left[2^{-|k_v|\alpha} \sum_{|k_w| \geq |k_v|, w \in \text{Lea}_v \mathbb{T} \setminus \{v\}} 2^{-|k_w|\alpha} \right] \\ &\lesssim \prod_{v \in \mathbf{v}_l} 2^{-|k_v|\alpha |V(\text{Lea}_v \mathbb{T})|}. \end{aligned} \quad (4.23)$$

Second step.

More generally, let $B_s[\mathbf{k}]$ be the expression obtained by $\tilde{\mathcal{P}}$ -projecting

$$\left(\prod_{m=1}^{l-1} [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_m} \circ \text{Roo}_{\mathbf{v}_{m+1}}(\mathbb{T})}]_s \right) [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_l}(\mathbb{T})}]_s \left(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\text{Lea}_{\mathbf{v}_1}(\mathbb{T}))} \Gamma(\ell(\sigma(v)))) \right)$$

onto the sum of terms with some fixed value of the indices $\mathbf{k} = (k_{v_{1,1}}, \dots, k_{v_{1,J_1}})$. Then

$$\|B_s[\mathbf{k}]\|_{\infty} \lesssim \prod_{v \in \mathbf{v}_1} 2^{-|k_v|\alpha |V(\text{Lea}_v \mathbb{T})|} \quad (4.24)$$

(proof by induction on l).

Third step.

We define

$$A_s(x) := [\mathcal{R} \text{SkI}_{\text{Roo}(\mathbb{T})}]_x \left(\prod_{m=1}^{l-1} [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_m} \circ \text{Roo}_{\mathbf{v}_{m+1}}(\mathbb{T})}]_s \right) [\mathcal{R} \text{SkI}_{\text{Lea}_{\mathbf{v}_l}(\mathbb{T})}]_s \left(\tilde{\mathcal{P}}^+(\otimes_{v \in V(\mathbb{T})} \Gamma(\ell(\sigma(v)))) \right) \quad (4.25)$$

(see eq. (4.22)), so that $A_{ts} = A_s(t) - A_s(s)$, and show that $\sup_{s \in \mathbb{R}} \|x \mapsto A_s(x)\|_{B_{\infty,\infty}^\alpha} < \infty$. Note first (see Remark following Lemma 4.6) there is no infra-red divergence problem.

Let $V(Roo(\mathbb{T})) = \{w_1 < \dots < w_{max}\}$. Fix $s \in \mathbb{R}$ and $K \in \mathbb{Z}$. By definition, and by Lemma 4.5,

$$(D(\phi_K)A_s)(x) = D(\phi_K) \left(x \mapsto \sum_{\mathbf{k}=(k_{v_{1,1}}, \dots, k_{v_{1,J_1}})} ((k_w)_{w \in V(Roo(\mathbb{T}))}) \in S_{\mathbf{k}} \int_{\prod_{v \in V(Roo(\mathbb{T}))} \text{supp}(\phi_{k_v})} \prod_{v \in V(Roo(\mathbb{T}))} d\xi_v \cdot e^{ix \sum_{v \in V(Roo(\mathbb{T}))} \xi_v} \frac{\prod_{w \in V(Roo(\mathbb{T}))} \mathcal{F}(D(\phi_{k_w})\Gamma'(\ell(\sigma(w))))(\xi_w)}{\prod_{w \in V(Roo(\mathbb{T}))} (\xi_w + \sum_{w' \rightarrow w, w' \in V(Roo(\mathbb{T}))} \xi_{w'})} B_s[\mathbf{k}] \right) \quad (4.26)$$

where indices in $S_{\mathbf{k}}$ satisfy in particular the following conditions:

- (i) $|\xi_w + \sum_{w' \rightarrow w, w' \in V(Roo(\mathbb{T}))} \xi_{w'}| > \frac{1}{2} \max\{|\xi_{w'}| : w' \rightarrow w, w' \in V(Roo(\mathbb{T}))\}$ by Lemma 4.4;
- (ii) $\left(\sum_{w \in V(Roo(\mathbb{T}))} \text{supp}(\phi_{k_w})\right) \cap (\text{supp}(\phi_K^*)) \neq \emptyset$, see eq. (4.16);
- (iii) for every $w \in V(Roo(\mathbb{T}))$, $|k_w| \leq |k_{w_{max}}|$; and
- (iv) for every $w \in V(Roo(\mathbb{T}))$, $|k_w| \leq |k_v|$ for every $v \in R(w) := \{v = v_{1,1}, \dots, v_{1,J_1} \mid v \rightarrow w\}$. Note that $R(w)$ may be empty. See Fig. 6.

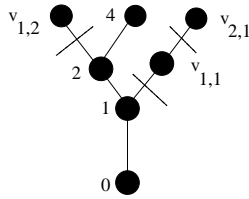


Figure 6: Here $V(Roo(\mathbb{T})) = \{0, 1, 2, 4\}$, $R(0) = R(4) = \emptyset$, $R(1) = \{v_{1,1}\}$, $R(2) = \{v_{1,2}\}$.

Note that $|k_{w_{max}} - K| = O(\log_2 |V(Roo(\mathbb{T}))|)$ by (ii) (see eq. (4.17)). Hence conditions (ii) and (iii) above are more or less equivalent to fixing $k_{w_{max}} \simeq K$ and letting $(k_w)_{w \in V(Roo(\mathbb{T})) \setminus \{w_{max}\}}$ range over some subset of $[-|K|, |K|] \times \dots \times [-|K|, |K|]$. The large fraction in eq. (4.26) contributes to $\|D(\phi_k)A_s\|_\infty$ an overall factor bounded by $|\Theta_1(\mathbf{k})| \prod_{w \in V(Roo(\mathbb{T}))} 2^{-|k_w|^\alpha}$.

If $w \in Roo(\mathbb{T})$, split $R(w)$ into $R(w)_> \cup R(w)_<$, where $R(w)_\geq := \{v \in R(w) \mid v \geq w_{max}\}$. Summing over indices corresponding to vertices in or above $R\mathbb{T}_> := \{v = v_{l,1}, \dots, v_{l,J_l} \mid v > w_{max}\} = \cup_{w \in Roo(\mathbb{T})} R(w)_>$, one gets by eq. (4.24) a quantity bounded up to a constant by

$$\prod_{v \in R\mathbb{T}_>} \sum_{|k_v| \geq |K|} 2^{-|k_v|\alpha|V(R_v\mathbb{T})|} \lesssim 2^{-|K|\alpha \sum_{v \in R\mathbb{T}_>} |V(R_v\mathbb{T})|}. \quad (4.27)$$

Let $w \in Roo(\mathbb{T}) \setminus \{w_{max}\}$ such that $R(w)_< \neq \emptyset$ (note that $R(w_{max})_< = \emptyset$). Let $R(w)_< = \{v_{i_1} < \dots < v_{i_j}\}$. Then the sum over (k_v) , $v \in R(w)_<$ contributes a factor bounded by a constant times

$$\begin{aligned} & 2^{-|k_w|\alpha} \sum_{|k_{v_{i_1}}|=|k_w|}^{\infty} \sum_{|k_{v_{i_2}}|=|k_{v_{i_1}}|}^{\infty} \dots \sum_{|k_{v_{i_j}}|=|k_{v_{i_{j-1}}}|}^{\infty} \\ & 2^{-|k_{v_{i_1}}|\alpha|V(Lea_{v_{i_1}}\mathbb{T})|} \dots 2^{-|k_{v_{i_j}}|\alpha|V(Lea_{v_{i_j}}\mathbb{T})|} \lesssim 2^{-|k_w|\alpha(1+\sum_{v \in R(w)_<} |V(Lea_v\mathbb{T})|)}. \end{aligned} \quad (4.28)$$

In other words, each vertex $w \in Roo(\mathbb{T})$ 'behaves' as if it had a weight $1 + \sum_{v \in R(w)_<} |V(R_v\mathbb{T})|$. Hence (by the same method as in the proof of Lemma 4.6), letting $R\mathbb{T}_< := \cup_{w \in Roo(\mathbb{T})} R(w)_<$,

$$\begin{aligned} \|D(\phi_K)A_s\|_{\infty} & \lesssim 2^{-|K|\alpha(|V(Roo(\mathbb{T}))|+\sum_{v \in R\mathbb{T}_<} |V(Lea_v\mathbb{T})|)} \cdot 2^{-|K|\alpha \sum_{v \in R\mathbb{T}_>} |V(Lea_v\mathbb{T})|} \\ & = 2^{-|K|\alpha|V(\mathbb{T})|}. \end{aligned} \quad (4.29)$$

□

5 Appendix. Hölder and Besov spaces

We gather in this Appendix some definitions and technical facts about Besov spaces and Hölder norms that are required in sections 2 and 4.

Definition 5.1 (Hölder norm) *If $f : \mathbb{R}^l \rightarrow \mathbb{R}$ is α -Hölder continuous for some $\alpha \in (0, 1)$, we let*

$$\|f\|_{\mathcal{C}^\alpha} := \|f\|_{\infty} + \sup_{x, y \in \mathbb{R}^l} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \quad (5.1)$$

The space $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^l)$ of real-valued α -Hölder continuous functions, provided with the above norm $\|\cdot\|_{\mathcal{C}^\alpha}$, is a Banach space.

Proposition 5.2 [30]

Let $l \geq 1$. There exists a family of C^∞ functions $\phi_0, (\phi_{1,j})_{j=1,\dots,4^l-2^l} : \mathbb{R}^l \rightarrow [0, 1]$, satisfying the following conditions:

1. $\text{supp}\phi_0 \subset [-2, 2]^l$ and $\phi_0|_{[-1,1]^l} \equiv 1$.
2. Cut $[-2, 2]^l$ into 4^l equal hypercubes of volume 1, and remove the 2^l hypercubes included in $[-1, 1]^l$. Let $K_1, \dots, K_{4^l-2^l}$ be an arbitrary enumeration of the remaining hypercubes, and $\tilde{K}_j \supset K_j$ be the hypercube with the same center as K_j , but with edges twice longer. Then $\text{supp}\phi_{1,j} \subset \tilde{K}_j$, $j = 1, \dots, 4^l - 2^l$.
3. Let $(\phi_{k,j})_{k \geq 2, j=1,\dots,4^l-2^l}$ be the family of dyadic dilatations of $(\phi_{1,j})$, namely,

$$\phi_{k,j}(\xi_1, \dots, \xi_l) := \phi_{1,j}(2^{1-k}\xi_1, \dots, 2^{1-k}\xi_l). \quad (5.2)$$

Then $(\phi_0, (\phi_{k,j})_{k \geq 1, j=1,\dots,4^l-2^l})$ is a partition of unity subordinated to the covering $[-2, 2]^l \cup \left(\bigcup_{k \geq 1} \bigcup_{j=1}^{4^l-2^l} 2^{k-1}\tilde{K}_j \right)$, namely,

$$\phi_0 + \sum_{k \geq 1} \sum_{j=1}^{4^l-2^l} \phi_{k,j} \equiv 1. \quad (5.3)$$

Constructed in this almost canonical way, the family of Fourier multipliers $(\phi_0, (\phi_{k,j}))$ is immediately seen to be uniformly bounded for the norm $\|\cdot\|_{S^0(\mathbb{R}^l)}$ defined in Proposition 5.8 below.

If $l = 1$, letting $K_1 = [1, 2]$ and $K_2 = [-2, -1]$, we shall write ϕ_1 , resp. ϕ_{-1} , instead of $\phi_{1,1}$, resp. $\phi_{1,2}$, and define $\phi_k(\xi) = \phi_{\text{sgn}(k)}(2^{1-|k|}\xi)$ for $|k| \geq 2$, so that $\sum_{k \in \mathbb{Z}} \phi_k \equiv 1$ and

$$\text{supp}\phi_0 \subset [-2, 2], \quad \text{supp}\phi_k \subset [2^{k-1}, 5 \times 2^{k-1}], \quad \text{supp}\phi_{-k} \subset [-5 \times 2^{k-1}, -2^{k-1}] \quad (k \geq 1). \quad (5.4)$$

In this particular case, such a family is easily constructed from an arbitrary even, smooth function $\phi_0 : \mathbb{R} \rightarrow [0, 1]$ with the correct support by setting $\phi_k(\xi) = \mathbf{1}_{\mathbb{R}_+}(\xi) \cdot (\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi))$ and $\phi_{-k}(\xi) = \mathbf{1}_{\mathbb{R}_-}(\xi) \cdot (\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi))$ for every $k \geq 1$ (see [31], §1.3.3).

In order to avoid setting apart the one-dimensional case, we let $\mathbb{I}_l := \mathbb{Z}$ if $l = 1$, and $\mathbb{I}_l = \{0\} \cup \{(k, j) \mid k \geq 1, 1 \leq j \leq 4^l - 2^l\}$ if $l \geq 2$. Also, if $l \geq 2$, we define $|\kappa| = k \geq 1$ if $\kappa = (k, j)$ with $k \geq 1$.

Definition 5.3 Let $(\tilde{\phi}_\kappa)_{\kappa \in \mathbb{I}_l}$ be the partition of unity of \mathbb{R}^l , $l \geq 1$ defined by (see Proposition 5.2) :

$$(i) \quad \tilde{\phi}_0 := \mathbf{1}_{[-1,1]^l}, \quad \tilde{\phi}_{1,j} := \mathbf{1}_{K_j}; \quad (5.5)$$

$$(ii) \text{ if } k \geq 2, \quad \tilde{\phi}_{k,j}(\xi_1, \dots, \xi_l) := \tilde{\phi}_{1,j}(2^{1-k}\xi_1, \dots, 2^{1-k}\xi_l). \quad (5.6)$$

We use this auxiliary partition several times in the text.

Definition 5.4 [30]

Let $\ell_\infty(L_\infty)$ be the space of sequences $(f_\kappa)_{\kappa \in \mathbb{I}_l}$ of a.s. bounded functions $f_\kappa \in L_\infty(\mathbb{R}^l)$ such that

$$\|f_\kappa\|_{\ell_\infty(L_\infty)} := \sup_{\kappa \in \mathbb{I}_l} \|f_\kappa\|_\infty < \infty. \quad (5.7)$$

Let $\mathcal{S}'(\mathbb{R}^l, \mathbb{R})$ be the dual of the Schwartz space of rapidly decreasing functions on \mathbb{R}^l . As well-known, it includes the space of infinitely differentiable slowly growing functions.

The following definition is classical. Recall that the Fourier transform \mathcal{F} has been defined at the end of the Introduction.

Definition 5.5 (Fourier multipliers) Let $m : \mathbb{R}^l \rightarrow \mathbb{R}$ be an infinitely differentiable slowly growing function. Then

$$D(m) : \mathcal{S}'(\mathbb{R}^l, \mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R}^l, \mathbb{R}), \quad \phi \mapsto \mathcal{F}^{-1}(m \cdot \mathcal{F}\phi) \quad (5.8)$$

defines a continuous operator.

In other words, m is a Fourier multiplier of $\mathcal{S}'(\mathbb{R}^l, \mathbb{R})$.

Definition 5.6 [30]

Let $B_{\infty,\infty}^\alpha(\mathbb{R}^l) := \{f \in \mathcal{S}'(\mathbb{R}^l, \mathbb{R}) \mid \|f\|_{B_{\infty,\infty}^\alpha} < \infty\}$ where

$$\begin{aligned} \|f\|_{B_{\infty,\infty}^\alpha} &:= \|2^{\alpha|\kappa|} D(\phi_\kappa) f\|_{\ell_\infty(L_\infty)} \\ &= \sup_{\kappa \in \mathbb{I}_l} 2^{\alpha|\kappa|} \|D(\phi_\kappa) f\|_\infty. \end{aligned} \quad (5.9)$$

Proposition 5.7 (see [30], §2.2.9)

For every $\alpha \in (0, 1)$, $B_{\infty,\infty}^\alpha(\mathbb{R}^l) = \mathcal{C}^\alpha(\mathbb{R}^l)$, and the two norms $\|\cdot\|_{\mathcal{C}^\alpha}$ and $\|\cdot\|_{B_{\infty,\infty}^\alpha}$ are equivalent.

We shall sometimes call $\|\cdot\|_{B_{\infty,\infty}^\alpha}$ the *Hölder-Besov norm*.

Let us finally give a criterion for a function m to be a Fourier multipliers of the Besov space $B_{\infty,\infty}^\alpha$:

Proposition 5.8 (Fourier multipliers) (see [30], §2.1.3, p.30)

Let $\alpha \in (0, 1)$ and $m : \mathbb{R}^l \rightarrow \mathbb{R}$ be an infinitely differentiable function such that

$$\|m\|_{S^0(\mathbb{R}^l)} := \sup_{|j| \leq l+5} \sup_{\xi \in \mathbb{R}^l} |(1 + \|\xi\|)^{|j|} m^{(j)}(\xi)| < \infty \quad (5.10)$$

where $j = (j_1, \dots, j_l)$, $|j| = j_1 + \dots + j_l$ and $m^{(j)} := \partial_{\xi_1}^{j_1} \dots \partial_{\xi_l}^{j_l} m$. Then there exists a constant C depending only on α , such that

$$\|D(m)f\|_{B_{\infty,\infty}^\alpha} \leq C \|m\|_{S^0(\mathbb{R}^l)} \|f\|_{B_{\infty,\infty}^\alpha}. \quad (5.11)$$

The space $S^0(\mathbb{R}^l)$ contains the space of translation-invariant pseudo-differential symbols of order 0 (see for instance [2], Definition 1.1, or [29]).

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